

On the Caginalp for a Conserved Phase-Field with Two Temperatures

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Abstract: The general theme of this article is the theoretical study of phase field systems, more precisely that of Caginalp. This work is motivated by their immense applications in many physical fields, industriels... The Caginalp problem gives the authors a formulation based on the fact that the phases separated by an unknown regular interface, which evolves in a regular way. The authors' aim in this paper is to study on Caginalp for a conserved Phase-field with two temperatures. The authors have worked on the existence and uniqueness of the Caginalp phase field in a conservative version. Moreover, the authors have also used Dirichlet type boundary conditions with a regular potential; existence and uniqueness are analyzed by means of absorbing bounded sets. The authors build the solution of the conservative problem on the estimates which lead authors to treat the problem well to arrive at the result. These equations are known as the conserved phase-field based on type II heat conduction and two temperatures. The authors consider a regular potential, more precisely a polynomial with edge conditions of Dirichlet type. More precisely, the authors prove the existence and uniqueness of solutions.

Keywords: A Conserved Phase-Field, Two Temperatures, Dirichlet Boundary Conditions, Regular Potential

1. Introduction

The Caginalp proposed two phase-field system in [10, 12], namely

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta \quad (1)$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t} \quad (2)$$

Called no conserved system, and

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \theta \quad (3)$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t} \quad (4)$$

Called no conserved system (in the sense that, when

endowed with Dirichlet boundary conditions, the spatial average of u is the order parameter, T is the relative temperature (defined as $T = \bar{T} - T_E$, where \bar{T} is the absolute temperature and T_E is the derivative of a double-well potential F (a typical choice is $F(s) = \frac{1}{4}(s^2 - 1)^2$, hence the usual cubic nonlinear term $f(s) = s^3 - s$ furthermore, we have set all physical parameter equal to one. These systems have been introduced to model phase transition phenomena, such as melting-solidification phenomena, and have been much studied from a mathematic point of view. Refer (total Ginzburg-landau) free energy

$$\Psi(u, \theta) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + f(u) - u\theta - \frac{1}{2} \theta^2 \right) dx \quad (5)$$

Where Ω is the domain occupied by the system (we assume here that it is a bounded and regular domain of R^n , $n=2$ or 3 , with boundary Γ) and the enthalpy.

$$H = u + T \quad (6)$$

$$q = -\nabla T \quad (15)$$

As far as the evolution equation for the order parameters is concerned, one postulates the relaxation dynamics (with relaxation parameter set equal to one)

$$\frac{\partial u}{\partial t} = -\frac{\partial \Psi}{\partial u} \quad (7)$$

Where $\frac{D}{Du}$ denotes a variational derivate with respect to u , which yields (1). Then, we have the energy equation

$$\frac{\partial H}{\partial t} = -\text{div} q \quad (8)$$

Where q is the heat flux. Assuming finally the usual fourier law for heat condition.

$$q = -\nabla T \quad (9)$$

We obtain (2) Now, one essential drawback of the Fourier law is that it predicts that thermal signals propagate at an infinite speed, which violates causality (the so-called paradox of heat conduction) To overcome this drawback, or at least to account for more realistic features, several alternative to the Fourier law, base, e.g., on the Max-well-cattaneo law or recent laws from thermo mechanics, in [8, 14, 16, 22].

In the late 1960s, several authors proposed a heat conduction theory based on two temperatures (see [16]). More precisely, one now considers the temperature T and the thermo dynamic temperature θ . In particular, for simple materials, these two temperature are shown to coincide. However, for non-simple materials, they differ and are related as follows

$$\theta = -\Delta T \quad (10)$$

The Caginalp system, based on this two temperature theory and the usual Fourier law, was studied in [17].

Our aim in this paper is to study a variant of the caginalp phase-field system based on the type II thermomechanics theory (see [11]) with two temperature recently proposed in [16].

In that case, the free energy reads, in terms of the (relative) thermo mechanics temperature θ .

$$\Psi(u, \theta) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + f(u) - u\theta - \frac{1}{2} \theta^2 \right) dx \quad (11)$$

and (7) yields, in view of (10), the following evolution equation for the order parameter

$$\frac{\partial u}{\partial t} - \Delta u - f(u) = -\Delta T \quad (12)$$

Furthermore, the enthalpy now reads

$$H = u + \theta = u - \Delta T \quad (13)$$

Which yields, owing to (8), the energy equation

$$\frac{\partial T}{\partial t} - \Delta T - \text{div} q = -\frac{\partial u}{\partial t} \quad (14)$$

Finally, the heat flux is given, in the type II theory with two temperatures, by (see [1, 3, 14, 23])

Where

$$\alpha(t, x) = \int_0^t T(\tau, x) d\tau + \alpha_0(x) \quad (16)$$

is the conductive thermal displacement. Nothing that

$T = \frac{\partial \alpha}{\partial t}$, we finally deduce from (12) and (14)-(15) the following variant of the Caginalp phase-fields system (see [7]):

$$\frac{\partial u}{\partial t} - \Delta u - f(u) = \frac{\partial \alpha}{\partial t} \quad (17)$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \quad (18)$$

In this paper, we conder the following conserved phase-field model:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \frac{\partial \alpha}{\partial t} \quad (19)$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \quad (20)$$

These equation are known as the conserved phase-fields system model (see [2, 5, 19, 22]) based on type II heat conduction and two temperatures [3, 4], conservative in the sens that, when endowed with Neumann boundary conditions, the spatial average of the order parameter is a conserved quantity. Indeed, in that case, integrating (19) over the spatial domain Ω , we have the conserved of mass

$$\langle u(t) \rangle = \langle u(0) \rangle \quad t \geq 0 \quad (21)$$

$$\langle . \rangle = \frac{1}{\text{vol} \Omega} \int_{\Omega} dx \quad (22)$$

Denotes the spatial average. Furthermore, integrating (20) over, obtain

$$\langle \alpha(t) \rangle = \langle \alpha(0) \rangle \quad t \geq 0 \quad (23)$$

Our aim in this paper is is to study the existence and uniqueness of solution of (17)-(18). We consider here only one the type boundary condition namely, Dirichlet (see [6, 18]). Furthermore, we consider regular term f (a usual choice being the cubic term $f(s) = s^3 - s$).

We consider the following initial and boundary value problem

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \frac{\partial \alpha}{\partial t} \quad (24)$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \quad (25)$$

$$u|_{\Gamma} = \Delta u|_{\Gamma} = \alpha|_{\Gamma} = 0, \text{ on } \Gamma \quad (26)$$

$$u|_{t=0} = u_0, \alpha|_{t=0} = \alpha_0, \frac{\partial \alpha}{\partial t} = \alpha_1 \quad (27)$$

As far as the nonlinear term f is concerned, we assume that

$$f \in C^2(R), f(0) = 0 \quad (28)$$

$$f'(s) \geq -C_0, C_0 \geq 0, s \in R \quad (29)$$

$$f(s)s \geq C_1 F(s) - C_2 \geq -C_3, C_1 > 0, C_2, C_3 \geq 0$$

$$s \in R \quad (30)$$

Where $F(s) = \int_0^s f(\xi) d\xi$. In particular, the usual cubic non linear term $f(s) = s^3 - s$ satisfies these assumptions.

Remark 2.1. We take here, for simplicity, Dirichlet boundary conditions. However, we obtain the same results for Neumann Boundary condition namely,

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = \frac{\partial \varphi}{\partial \nu} \text{ on } \Gamma \quad (31)$$

Where ν denotes the unit outer normal to Γ . To do so, we rewrite, owing to (1) and (7), the equations in the form

$$\frac{\partial \bar{u}}{\partial t} + \Delta^2 \bar{u} - \Delta(f(u) - \langle f(u) \rangle) = -\Delta \frac{\partial \bar{\alpha}}{\partial t}$$

$$\frac{\partial^2 \bar{\varphi}}{\partial t^2} - \Delta \frac{\partial \bar{\varphi}}{\partial t} - \Delta \bar{\varphi} = -\frac{\partial \bar{u}}{\partial t}$$

Where $\bar{\nu} = \nu - \langle \nu \rangle$, $|\langle \nu_0 \rangle| \leq M_1$, $|\langle \nu_0 \rangle| \leq M_2$. For fixed positive constants M_1 and M_2 . Then, we note that

$$\nu \rightarrow \left(\|(-\Delta)^{-1/2} \nu\| + \langle \nu \rangle^2 \right)^2$$

Where, here $-\Delta$ denotes the minus Laplace operators with Neumann Boundary conditions and action on functions with null average where it is understood that

$$\langle \cdot \rangle = \frac{1}{\text{vol}(\Omega)} \langle \cdot, 1 \rangle_{H^{-1}(\Omega) H^{-1}(\Omega)},$$

Furthermore

$$(-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta u - f(u) = \frac{\partial \alpha}{\partial t} \quad (33)$$

We multiply (33) by $\frac{\partial u}{\partial t}$ and integrating over Ω . We have

$$\frac{d}{dt} \left(\|\nabla u\|^2 + 2 \int_{\Omega} F(u) dx \right) + 2 \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 = 2 \left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right) \quad (34)$$

Then we multiply (25) by $\frac{\partial \alpha}{\partial t}$ and integrating over Ω . We obtain

$$\frac{d}{dt} \left(\|\nabla \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) + 2 \left\| \frac{\partial \alpha}{\partial t} \right\|^2 = -2 \left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right) \quad (35)$$

Summing (34) and (35), we find the differential inequality of the form

$$\frac{d}{dt} E_1 + C \left(\left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 \right) \leq C', C > 0 \quad (36)$$

where

$$E_1 = \|\nabla u\|^2 + 2 \int_{\Omega} F(u) dx + \|\nabla \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \quad (37)$$

Satisfies

$$E_1 \geq \left(\|\nabla u\|^2 + 2 \int_{\Omega} F(u) dx + \|\nabla \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) - C'' \quad (38)$$

Hence estimates on $u, \alpha \in L^\infty(0, T; H_0^1(\Omega))$,

$$\nu \rightarrow \left(\|\bar{\nu}\| + \langle \nu \rangle^2 \right)^2$$

$$\nu \rightarrow \left(\|\nabla \nu\| + \langle \nu \rangle^2 \right)^2$$

$$\nu \rightarrow \left(\|\Delta \nu\| + \langle \nu \rangle^2 \right)^2$$

are norms in $H^{-1}(\Omega)$, $L^2(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$, respectively, which are equivalent to the usual ones.

We further assume that

$$|f(s)| \leq \varepsilon F(s) + C_\varepsilon, \varepsilon > 0, s \in R \quad (32)$$

Which allows to deal with term $\langle f(u) \rangle$.

2. Notation

We denotes by $\|\cdot\|$ the usual L^2 -norm (with associated product scalar (\cdot, \cdot)) and set $\|\cdot\|_{-1} = \|(-\Delta)^{-\frac{1}{2}} \cdot\|$, where $-\Delta$ denotes the minus Laplace operator with Dirichlet Boundary conditions. More generally, $\|\cdot\|_X$ denotes the norm of Banach space X .

Throughout this paper, the same letters C_1, C_2 and C_3 denote (generally positives) constants which may change from line and line, or even a same line.

3. A Priori Estimate

The estimates derived in this subsection will be formal, but they can easily be justified with in a Galerkin scheme. We rewrite (24) in the equivalent form.

$$\frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega)) \text{ and}$$

$$\frac{\partial \alpha}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$$

We multiply (33) by $-\Delta \frac{\partial u}{\partial t}$ and integrating over Ω . We obtain

$$\frac{d}{dt} (||\nabla u||^2) + 2 ||\frac{\partial u}{\partial t}||^2 = -2 \left(\frac{\partial u}{\partial t}, \Delta \frac{\partial \alpha}{\partial t} \right) + \left(\Delta f(u), \frac{\partial u}{\partial t} \right) \quad (39)$$

Which yields to (28) and the continuous embedding $H^2(\Omega) \subset C(\overline{\Omega})$

$$\frac{d}{dt} (||\nabla u||^2) + 2 ||\frac{\partial u}{\partial t}||^2 \leq Q(||u||_{H_2}) - 2 \left(\frac{\partial u}{\partial t}, \Delta \frac{\partial \alpha}{\partial t} \right) \quad (40)$$

Then we multiply (25) by $-\Delta \frac{\partial \alpha}{\partial t}$ and integrating over Ω . We obtain

$$\left(\frac{\partial^2 \alpha}{\partial t^2}, -\Delta \frac{\partial \alpha}{\partial t} \right) + ||\Delta \frac{\partial \alpha}{\partial t}||^2 + \frac{1}{2} \frac{d}{dt} ||\Delta u||^2 = \left(\frac{\partial u}{\partial t}, \Delta \frac{\partial \alpha}{\partial t} \right)$$

Which give

$$\frac{d}{dt} (||\Delta \alpha||^2 + ||\nabla \frac{\partial \alpha}{\partial t}||^2) + 2 ||\Delta \frac{\partial \alpha}{\partial t}||^2 = -2 \left(\frac{\partial u}{\partial t}, \Delta \frac{\partial \alpha}{\partial t} \right) \quad (41)$$

Summing then (40) and (41), we obtain

$$\frac{d}{dt} (||\Delta u||^2 + ||\Delta \alpha||^2 + ||\nabla \frac{\partial \alpha}{\partial t}||^2) + 2 ||\Delta \frac{\partial \alpha}{\partial t}||^2 + ||\frac{\partial u}{\partial t}||^2 \leq Q(||u||_{H_2}) \quad (42)$$

In particular, setting

$$y = ||\Delta u||^2 + ||\Delta \alpha||^2 + ||\nabla \frac{\partial \alpha}{\partial t}||^2$$

We deduce from (42) in equation of the form

$$y' \leq Q(y) \quad (43)$$

Let z be the solution to the ordinary differential equation

$$Z' \leq Q(Z), Z(0) = y(0) \quad (44)$$

It following the comparaisn principal that there exist

$T_0 = T_0(||u_0||_{H_2}, ||\alpha_0||_{H_2}, ||\alpha_1||_{H_1})$ belonging to say, $(0, \frac{1}{2})$ such that

$$y(t) \leq Z(t), \text{ pour tout } t \in [0, T] \quad (45)$$

hence

$$||u||_{L^2(\Omega)}^2 + ||\alpha(t)||_{H^2(\Omega)}^2 + ||\frac{\partial \alpha}{\partial t}||_{H^1(\Omega)}^2 \leq Q(||u_0||_{H_2}, ||\alpha_0||_{H_2}, ||\alpha_1||_{H_1}), t \leq T \quad (46)$$

We now differentiate (31) with respect to time and have, noting that

$$\frac{\partial^2 \alpha}{\partial t^2} = \Delta \frac{\partial \alpha}{\partial t} + \Delta \alpha - \frac{\partial u}{\partial t} \quad (47)$$

We have

$$(-\Delta)^{-1} \frac{\partial}{\partial t} \frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial t} - f'(u) \frac{\partial u}{\partial t} = \Delta \frac{\partial \alpha}{\partial t} + \Delta \alpha - \frac{\partial u}{\partial t} \quad (48)$$

We mulply (48) by $t \frac{\partial u}{\partial t}$ and we find, owing to (25)

$$\frac{d}{dt} \left(t ||\frac{\partial u}{\partial t}||_{-1}^2 \right) + \frac{t}{2} ||\nabla \frac{\partial u}{\partial t}||^2 \leq Ct \left(||\frac{\partial u}{\partial t}||_{-1}^2 + ||\Delta \alpha||^2 \right) + ||\Delta \frac{\partial \alpha}{\partial t}||^2 \quad (49)$$

Which yields employing the interpolation inequations

$$\left\| \frac{\partial u}{\partial t} \right\|^2 \leq \left\| \frac{\partial u}{\partial t} \right\|_{-1} \left\| \nabla \frac{\partial u}{\partial t} \right\|^2$$

In particular, deduce (36), (46) and (49), applying the Gronwall's lemma that

$$\left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 \leq Q(\|u_0\|_{H_2}, \|\alpha_0\|_{H_2}, \|\alpha_1\|_{H_1}), t \in [0; T_0] \quad (50)$$

Multiplying then by (33), we have, proceeding as above

$$\frac{d}{dt} \left(\left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 \right) + \left\| \nabla \frac{\partial u}{\partial t} \right\|^2 \leq C \left(\left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \|\Delta \alpha\|^2 \right) + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \quad (51)$$

It thus follows from (36), (51) and Gronwall's lemma that

$$\left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 \leq e^c Q(\|u_0\|_{H_2}, \|\alpha_0\|_{H_2}, \|\alpha_1\|_{H_1}) \left\| \frac{\partial u(T_0)}{\partial t} \right\|_{-1}^2, \quad t \geq T_0 \quad (52)$$

Hence, owing to (50)

$$\left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 \leq e^c Q(\|u_0\|_{H_2}, \|\alpha_0\|_{H_2}, \|\alpha_1\|_{H_1}), \quad t \geq T_0 \quad (53)$$

We rewrite (36) in the form

$$\Delta u + f(u) = h_u(t), u = \text{oon } \Gamma \quad (54)$$

For $t \geq T_0$ fixed, here

$$h_u(t) = (-\Delta)^{-1} \frac{\partial u}{\partial t} + \frac{\partial \alpha}{\partial t} \quad (55)$$

Satisfies, owing to (36) and (53)

$$\|h_u(t)\| \leq e^c Q(\|u_0\|_{H_2}, \|\alpha_0\|_{H_2}, \|\alpha_1\|_{H_1}), \quad t \geq T_0 \quad (56)$$

We multiply (54) by u and have, noting that

$$\begin{aligned} f'(s) &\geq -C_0, C_0 \geq 0, s \in R \\ \|\nabla u\|^2 &\leq c \|h_u(t)\| + c \end{aligned} \quad (57)$$

Then we multiply (54) by $-\Delta u$ and we find, owing to (29)

$$\|\Delta u\|^2 \leq c(\|h_u(t)\|^2 + \|\nabla u\|^2) \quad (58)$$

We thus deduce from (56)-(58) that

$$\|u\|_{H^2}^2 \leq e^c Q(\|u_0\|_{H_2}, \|\alpha_0\|_{H_2}, \|\alpha_1\|_{H_1}) t \geq T_0 \quad (59)$$

And thus, owing to (46)

$$\|u\|_{H^2}^2 \leq e^c Q(\|u_0\|_{H_2}, \|\alpha_0\|_{H_2}, \|\alpha_1\|_{H_1}) t \geq T_0 \quad (60)$$

Returning to (41), have

$$\frac{d}{dt} \left(\|\Delta \alpha\|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 \right) + \left\| \Delta \frac{\partial \alpha}{\partial t} \right\|^2 \leq \left\| \frac{\partial u}{\partial t} \right\|^2 \quad (61)$$

Noting that it follows (36), (51) and (53) that

$$\int_{T_0}^t \left(\|\Delta \alpha\|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 \right) d\tau \leq e^c Q(\|u_0\|_{H_2}, \|\alpha_0\|_{H_2}, \|\alpha_1\|_{H_1}) t \geq T_0 \quad (62)$$

We finally deduce from (46) and (60)-(62) that

$$\|u\|_{H^2(\Omega)}^2 + \|\alpha(t)\|_{H^2(\Omega)}^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|_{H^2(\Omega)}^2 \leq Q(\|u_0\|_{H_2}, \|\alpha_0\|_{H_2}, \|\alpha_1\|_{H_1}), \quad t \geq 0 \quad (63)$$

4. Existence and Uniqueness of Solution

Theorem 5.1. We assume (28)-(30) and $(u_0, \alpha_0, \alpha_1) \in$

$$(H^2(\Omega) \cap H^1(\Omega)) \times (H^2(\Omega) \cap H^1(\Omega)) \times (H^1(\Omega))$$

Then (24)-(27) possesses at last one solution $(u, \alpha, \frac{\partial \alpha}{\partial t})$

Such that $u, \alpha \in L^\infty(0, T; H^2(\Omega) \cap H^1(\Omega)), \frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega))$ and

$$\frac{\partial \alpha}{\partial t} \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H^1(\Omega))$$

Proof The proof is based on (63) and e.g., a standard Galerkin scheme. We have concerning the uniqueness, the following.

Theorem 5.2. We assume that the assumptions of the theorem 4.1. hold. Then, the solution obtained in theorem 4.1. is unique.

Let $(u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t})$ and $(u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t})$ be two solutions (24)-(27) with initial data $(u_0^1, \alpha_0^1, \alpha_1^1)$ and $(u_0^2, \alpha_0^2, \alpha_1^2)$, respectively. We set

$$(u, \alpha, \frac{\partial \alpha}{\partial t}) = (u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}) - (u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t})$$

and

$$(u_0, \alpha_0, \alpha_1) = (u_0^1, \alpha_0^1, \alpha_1^1) - (u_0^2, \alpha_0^2, \alpha_1^2)$$

Then, (u, α) satisfies

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u^1) - f(u^2)) = -\Delta \frac{\partial \alpha}{\partial t} \quad (64)$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \quad (65)$$

$$u|_\Gamma = \Delta u|_\Gamma = \alpha|_\Gamma = 0, \text{ on } \Gamma \quad (66)$$

$$u|_{t=0} = u_0, \alpha|_{t=0} = \alpha_0, \frac{\partial \alpha}{\partial t} = \alpha_1 \quad (67)$$

We multiply (33) by $(-\Delta)^{-1} \frac{\partial u}{\partial t}$ and integrating over Ω . We have

$$\frac{d}{dt} (\|\nabla u\|^2) + 2 \|\frac{\partial u}{\partial t}\|_{-1}^2 = 2 \left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right) - 2(f(u^1) - f(u^2), \frac{\partial u}{\partial t}) \quad (68)$$

We multiply (33) by $\frac{\partial \alpha}{\partial t}$ and integrating over Ω . We obtain

$$\frac{d}{dt} (\|\nabla \alpha\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2) + 2 \|\frac{\partial \alpha}{\partial t}\|^2 = -2 \left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right) \quad (69)$$

Summing (68) and (69), we find

$$\frac{d}{dt} \left(\|\nabla u\|^2 + \|\nabla \alpha\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 \right) + 2 \|\frac{\partial u}{\partial t}\|_{-1}^2 = 2 \left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right) - 2(f(u^1) - f(u^2), \frac{\partial u}{\partial t})$$

We know

$$2 \left| (f(u^1) - f(u^2), \frac{\partial u}{\partial t}) \right| \leq 2 \|(-\Delta)^{\frac{1}{2}}(f(u^1) - f(u^2))\| \|(-\Delta)^{-\frac{1}{2}} \frac{\partial u}{\partial t}\|$$

Which implies

$$2 \left| (f(u^1) - f(u^2), \frac{\partial u}{\partial t}) \right| \leq 2 \|(-\Delta)^{\frac{1}{2}}(f(u^1) - f(u^2))\|^2 + \|(-\Delta)^{-\frac{1}{2}} \frac{\partial u}{\partial t}\|^2$$

Which yields

$$2\left|f(u^1) - f(u^2), \frac{\partial u}{\partial t}\right| \leq Q(\|u\|) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2$$

Where, here and below

$$Q = Q(\|u_0^1\|_{H_2}, \|\alpha_0^1\|_{H_2}, \|\alpha_1^1\|_{H_1}, \|u_0^2\|_{H_2}, \|\alpha_0^2\|_{H_2}, \|\alpha_1^2\|_{H_1}),$$

Therefore

$$\frac{d}{dt} \left(\|\nabla u\|^2 + \|\nabla \alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 \right) + 2 \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + 2 \left\|\frac{\partial \alpha}{\partial t}\right\|^2 \leq Q(\|u\|) \quad (70)$$

In particular

$$\frac{d}{dt} \left(\|\nabla u\|^2 + \|\nabla \alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 \right) \leq Q(\|u\|) \quad (71)$$

It thus following form (71) and Gronwall's lemma that

$$\|u\|_{H^2(\Omega)}^2 + \|\alpha(t)\|_{H^2(\Omega)}^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|_{H^2(\Omega)}^2 \leq ce^t Q(\|u_0\|_{H_2}, \|\alpha_0\|_{H_2}, \|\alpha_1\|_{H_1}), t \geq 0 \quad (72)$$

Hence the uniqueness, as well continuous dependence with respect to the initial data.

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