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Axisymmetric Problem of Stationnary Navier-Stokes Equations Coupled with the Heat Equation

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Abstract: In this work, we are interested in the mathematical study of the flow of a Newtonian Navier-Stokes fluid, coupled to the energy equation, in a domain with axial symmetry. The study consists first of all in reducing this problem, which is posed in a domain in dimension three (3-D), to a problem whose spatial domain is in dimension two, using the transformation of Cartesian coordinates in cylindrical coordinates, assuming that the problem data does not depend on the angle of rotation. The problem thus obtained is a so-called axially symmetric problem presenting a degeneracy on the axis of symmetry, hence the interest of this study. The study of this problem is the subject of the first part of this article which deals with the existence and uniqueness of the weak solution of the problem in a Sobolev space with appropriate weight. The results of this part have already been published by the same authors that we recall here with some slight modifications in order to facilitate the reading and understanding of the second part of the article. In this second part, we approach the existence and the unicity of the numerical solution of the posed problem. It is obtained using the Lagrange finite element method whose polinomial space is of degree one. The study in question highlights the necessary algebraic relations between the different physical parameters of the problem to which the flow in question obeys.

Keywords: Newtonian Fluid, Navier-Stokes Equations, Axisymmetric Problem, Weak Solution, Numerical Solution, Finite Element Method, Weighted Sobolev Space

1. Introduction and Formulation of the Problem

1.1. Introduction

For essentially economic reasons, one of the most widely used technologies currently producing electricity on the *concentrated solar power principle* (CSP) is the *parabolic solar concentrator* (PTC). The physico-mathematical modeling of the different processes involved in this technical device has been the subject of intense studies by the specialized scientific community for several decades.

Recall that a solar parabolic concentrator is a technology used in a solar power plant to increase the thermal energy of a heat transfer fluid charged with transporting heat and circulating in a collector tube called *solar collector*. This fluid, so called is then pumped into conventional exchangers to produce superheated steam, called a thermodynamic fluid, which drives a turbine to generate electricity.

For the detailed description and operation of this device we refer the reader, for example, to works [5, 9, 12, 17] and to the abundant literature cited in these works.

In this work, we limit ourselves to the mathematical study (existence and uniqueness of the weak and the numerical solution) of stationary Navier-Stokes equations modeling the flow process of a viscous incompressible Newtonian fluid in the PTC collector tube and subjected to temperature-dependent gravity force, when this is a solution of the heat equation. The force of gravity intervening in the Navier-Stokes system is approximated by the Boussinesq method.

We will formulate the physical problem with respect to Cartesian coordinates (x, y, z). His mathematical study will be carried out with respect to the cylindrical coordinates (r, θ, z) in the case of an axisymetrical problem.

1.2. Formulation of the Physical Model

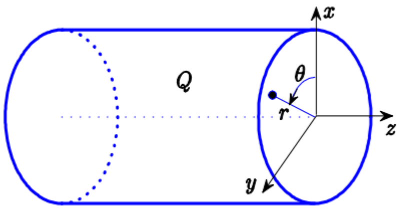

Figure 1. *Cylinder.*

The model presented here is far from modeling all aspects of the physical processes involved in a `PTC`-type device. We are interested here only in the velocity of the flow and the temperature generated in the fluid from a given source.

Let $Q \subset \mathbb{R}^3$ be an open cylindrical tube of the `PTC` and placed horizontally with respect to a Cartesian coordinates system (x, y, z) as shown in the Figure 1. We assume that Q is occupied by a Newtonian fluid, visqueus and incompressible such that:

1. the flow is completely developed turbulent;
2. the regime is stationary;
3. the weight of the fluid is the only body force acting on the fluid.

Note that a turbulence is a flow property and not a fluid property, which makes sense only in a three-dimensional space and occurs at high Reynolds numbers.

Let $\mathbf{u} = (u_x, u_y, u_z)^t$ be the velocity field of the through-flow in Q, where by $\mathbf{w}^t$ we denote the transpose of the vector (w_x, w_y, w_z). Then, taking into account the above assumptions, the flow of a viscous, Newtonian and incompressible fluid is governed by the following equations:

1) *Continuity equation:*

$$\nabla u \equiv \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0, \tag{1}$$

2) *Navier-Stokes equations in a vector form:*

$$-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \frac{1}{\rho_0} \nabla p = \mathcal{F}, \tag{2}$$

where, Δ and ∇ are the usual differential operators of Laplace and divergence respectively and $\mathcal{F} = (0, 0, \mathcal{F}_z)^t$.

The vector equation (2) in a scalar form can be written as

- flow along the x axis:

$$-\nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}, \tag{3}$$

- flow along the y axis:

$$-\nu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \right) + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial y}, \tag{4}$$

- flow along the z axis:

$$\begin{gathered} -\nu \left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right) \\ + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + \mathcal{F}_z, \end{gathered} \tag{5}$$

Where ν is the kinematic viscosity of the fluid, p is the pressure of the fluid and $\mathcal{F}_z$ is the gravity vector force.

Since Q is subjected to an external heat source, then assuming that its density depends only on the temperature T, and placing itself in the standard Boussinesq approximation, the gravitational force $\mathcal{F}_z$ can be written in the following form:

$$\mathcal{F}_z = \beta_0 T.$$

Here, $\beta_0 = \rho \beta g_0 / \rho_0$, where β is the thermal expansion coefficient of the fluid, g_0 is the gravity acceleration and ρ is the density of the fluid.

On the other hand, the temperature T satisfies the following equation:

3) *Energy equation:*

$$-\lambda \Delta T + (\mathbf{u}.\nabla)\, T = g, \tag{6}$$

where λ is the thermal diffusivity, g is a source term depending only on the domain Q.

To establish the equation (6), it is assumed that the flow has a very low Mach number, i.e. the ratio between the speed of flow and that of sound is very low.

Note that the coefficients λ, ν and β are evaluated at a reference temperature T_0 which is the free stream temperature.

We associate to the système $(1) - (6)$ the homogeneous boundary conditions of Dirichlet:

$$u|_{\partial Q} = T|_{\partial Q} = 0, \tag{7}$$

where ∂Q is the boundary of the domain Q.

The mathematical study of the problem $(1) - (7)$ has been the subject of many works, for a long time, see for example [2, 6, 16].

1.3. Formulation of the Axisymmetric Problem

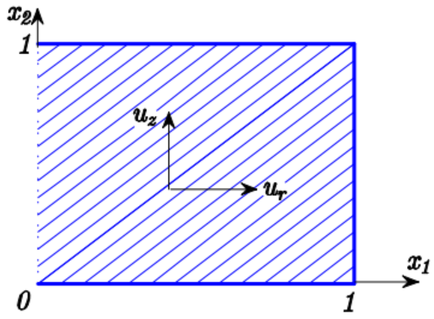

Figure 2. Cartesian coordinates.

In the following, instead of treating the problem $(1)-(7)$ in Cartesian coordinates, we will do it in cylindrical coordinates (r, θ, z), where $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $r > 0$, $0 \leq \theta < 2\pi$ (see the orientation of co-ordinates sytem on the Figure 1), while assuming that the fluid flow, coupled with the equation of energy, is axisymmetric flow, i.e. the components of the flow velocity, the pressure and the temperature will not depend on the angle of rotation θ. In this case, one speaks of an *axisymetric flux without swirl*. For more details about the relation between the swirl flux and the no swirl flux, see [4]. In Thus, the vector $\mathbf{u}$ of velocity of the fluid is reduced to two components, one is radial u_r, the other is axial u_z, i.e. $\mathbf{u} = (u_r, u_z)$ with respect the basis $\left(\overrightarrow{i_r}, \overrightarrow{k}\right)$, where

$$\overrightarrow{i_r} = (\cos\theta, \sin\theta, 0), \ \overrightarrow{k} = (0, 0, 1),$$

see [1]. Not to be confused turbulent flow and swirl flow.

Then, according this transformation and taking into account the fact that the component u_θ along the axis rotation is zero (no swirl flow), we obtain the following system (For calculating derivatives with respect to cylindrical coordinates we refer the reader to ([19], p.838):

1) Continuity equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0. \tag{8}$$

2) Navier-Stokes equations:

- flow along the r axis:

$$-\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2}\right] + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r} + \beta_0 T, \tag{9}$$

- flow along the z axis:

$$-\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2}\right] + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}. \tag{10}$$

3) Energy equation:

$$-\lambda\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] + u_r\frac{\partial T}{\partial r} + u_z\frac{\partial T}{\partial z} = g. \tag{11}$$

Let's put $r = x_1$, $z = x_2$, $u_r = u_1$, $u_z = u_2$, $T = u_3$, $\mathbf{u} = (u_1, u_2)^t$ and $\mathbf{U} = (u_1, u_2, u_3)^t$, where $(x_1, x_2) \in \Omega$, $\Omega =]0,1[\times]0,1[$, see Figure 2, where the axis (Oz) represents that of the cylinder Q in Figure 1.

Therefore, we can write the equations (8), $(9)-(10)$ and (11) in the following vector form:

$$-\Delta_r \mathbf{U} + (\mathbf{u}\cdot\nabla)\mathbf{U} = \mathbf{f}(p, u_3), \tag{12}$$

where

$$\Delta_r \mathbf{U} = (\nu\Delta_r \mathbf{u}, \lambda\Delta_r u_3)^t, \ \Delta_r \mathbf{u} = (\Delta_r u_1, \Delta_r u_2)^t,$$

$$\Delta_r u_1 = \frac{1}{x_1}D_1(x_1 D_1 u_1) + D_2^2 u_1 - \frac{u_1}{x_1^2}, \ D_i = \frac{\partial}{\partial x_i}\ (i = 1, 2);$$

$$\Delta_r u_j = \frac{1}{x_1}D_1(x_1 D_1 u_j) + D_2^2 u_j \ \ (j = 2, 3);$$

$$(\mathbf{u}\cdot\nabla)\mathbf{U} = \sum\nolimits_{i=1}^{2} u_i D_i \mathbf{U}, \ D_i\mathbf{U} = (D_i u_1, D_i u_2, D_i u_3);$$

$$\mathbf{f}(p, u_3) = (f_1(p, u_3), \ f_2(p), \ f_3)^t,$$

$$f_1(p, u_3) = \frac{\partial p}{\partial x_1} + \beta_0 u_3, \ f_2(p) = -\frac{\partial p}{\partial x_2}, \ f_3 = g,$$

equipped with the boundary conditions

$$\mathbf{U}|_\Gamma = 0, \ \Gamma = \partial\Omega\backslash\Gamma_0, \ \Gamma_0 = \{(0, x_2) : 0 < x_2 < 1\}. \tag{13}$$

Our goal is to establish the existence and uniqueness of the weak solution of the vectorial axisymetrical problem $(12) - (13)$ in an appropriate weighted sobolev space.

Unlike the approach of decoupling the field of flow velocities from temperature to study the existence and uniqueness of the weak solution of the Navier-Stokes problem coupled to the energy equation (3D), see [2, 6, 16], here we treat the corresponding axisymmetric problem (2D) in vectorial form. Of course, here, the study takes place in an appropriate weighted Sobolev space.

The mathematical study of the axisymmetric problem of the Navier-Stokes equations has been the subject of many works under different aspects, existence uniqueness and regularity of the weak solution of the problem, see for example [1, 4, 15, 20] and the references cited in these works.

In the case of an elliptical problem with axial symmetry, we refer the reader to the article [14].

2. Functional Space of Study

2.1. Weighted Sobolev Spaces

In this paragraph, we introduce some appropriate functional spaces in which the system $(12) - (13)$ will be studied.

Let $C_0^\infty\left(\overline{\Omega}\right)$ be the space of infinitely differentiable functions on $\overline{\Omega}$ equipped with the norm

$$\|u\|_{p,\mu} = \left(\int_\Omega |u|^p x_1 dx\right)^{1/p},$$

where $dx = dx_1 dx_2$. In the follow, instead of $x_1 dx$ we write $d\mu(x)$, where $\mu(\Omega) = \int_\Omega x_1 dx$ is a *density measure*.

Let $L_p(\Omega,\mu)$ be the completed of $C_0^\infty\left(\overline{\Omega}\right)$ with respect to the norm $\|u\|_{p,\mu}$, i.e. $L_p(\Omega,\mu) = \overline{C_0^\infty\left(\overline{\Omega}\right)}^{\|\cdot\|_{p,\mu}}$, see [8].

We denote by $W^{k,p}(\Omega,\mu)$, $1 \le p < \infty$, the so called *weighted Sobolev space* of all scalar functions $u = u(x)$ which are defined a.e. on Ω and whose generalized derivatives $D^\alpha u(x) \in L_p(\Omega,\mu)$ for all α such that $|\alpha| \le k$, where

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2}, |\alpha| = \alpha_1 + \alpha_2.$$

The space $W^{k,p}(\Omega,\mu)$ is a Banach space as completed space of $C_0^\infty\left(\overline{\Omega}\right)$ with respect to the norm

$$\|u\|_{k,p,\mu} = \left(\sum_{|\alpha|\le k}\int_\Omega |D^\alpha u(x)|^p d\mu(x)\right)^{1/p},$$

i.e $W^{k,p}(\Omega,\mu) = \overline{C_0^\infty\left(\overline{\Omega}\right)}^{\|\cdot\|_{k,p,\mu}}$, see also [10]. If $k = 0$, we write $W^{0,p}(\Omega,\mu) = L_p(\Omega,\mu)$.

The spaces $L_2(\Omega,\mu)$ and $W^{k,2}(\Omega,\mu)$ are the Hilbert spaces with respect to the scalar product:

$$(u,v) = \int_\Omega (uv)(x)\ d\mu(x),\ (u,v)_k = \sum_{|\alpha|\le k}(D^\alpha u, D^\alpha v),$$

respectivily.

In the case of the standard Sobolev space $W^{k,2}(\Omega)$, we write

$$\|u\|_{k,p} = \left(\sum_{|\alpha|\le k}\int_\Omega |D^\alpha u(x)|^p dx\right)^{1/p}.$$

We note E^m the $m-th$ cartesian product $\underbrace{E \times \ldots \times E}_{m-times}$ of any set E.

Consider the space:

$$C^1_{0,\mathbf{\Gamma}}\left(\overline{\Omega}\right) = \left\{v \in C^1\left(\overline{\Omega}\right) : v|_{\mathbf{\Gamma}} = 0\right\},$$

and denote

$$\mathcal{V} = \left\{\mathbf{v} = (v_1,v_2) \in \left(C^1_{0,\mathbf{\Gamma}}\left(\overline{\Omega}\right)\right)^2 : \nabla_r \mathbf{v} \equiv \tfrac{1}{x_1} D_1(x_1 v_1) + D_2 v_2 = 0\right\},$$

and $\mathcal{W} = \mathcal{V} \times C^1_{0,\mathbf{\Gamma}}\left(\overline{\Omega}\right)$.

By $\mathbb{V}$ and $\mathbb{W}$ we denote, respectively, the closure of $\mathcal{V}$ and $\mathcal{W}$ with respect to the norm

$$|\mathbf{V}|_{1,2,\mu} = \left(\sum_{j=1}^2 |v_j|^2_{1,2,\mu}\right)^{1/2} \text{ and } |\mathbf{V}|_{1,2,\mu} = \left(\sum_{j=1}^3 |v_j|^2_{1,2,\mu}\right)^{1/2}.$$

Sometimes, we denote the norm in $\mathbb{V}$ (resp. in W), by $\|\cdot\|_{\mathbb{V}}$ (resp. by $\|\cdot\|_{\mathbb{W}}$).

We introduce the following weighted subspace

$$W^{1,2}_{0,\mathbf{\Gamma}}(\Omega,\mu) = \left\{v \in W^{1,2}(\Omega,\mu) : v|_{\partial\Omega\setminus\mathbf{\Gamma}} = 0\right\} \subset W^{1,2}(\Omega,\mu),$$

equipped with the norm

$$|u|_{1,2,\mu} = \left(\sum_{i=1}^{2} \|D_i u\|_{2,\mu}^2 \right)^{1/2}.$$

Let's mention that the space $W_{0,\mathbf{\Gamma}}^{1,2}(\Omega,\mu)$ is the closure of $C_{0,\mathbf{\Gamma}}^1\left(\overline{\Omega}\right)$ with respect to the norm $|\cdot|_{1,2,\mu}$, i.e.

$$W_{0,\mathbf{\Gamma}}^{1,2}(\Omega,\mu) = \overline{C_{0,\mathbf{\Gamma}}^1\left(\overline{\Omega}\right)}^{|\cdot|_{k,p,\mu}}.$$

For more details on the weighted sobolev spaces we refer the reader to the monograph [13].

2.2. Some Prelimenary Results in the Space $W^{k,p}(\Omega,\mu)$

To establish the variational formulation of problem (12) and (13), we will need the following lemma:

Lemma 2.1. *Let* $u_i \in W_{0,\mathbf{\Gamma}}^{1,2}(\Omega,\mu)$, $i = 1,2,3$ *and* $v \in W_{0,\mathbf{\Gamma}}^{1,2}(\Omega,\mu) \cap C^1\left(\overline{\Omega}\right)$. *Then, there is a sequence* $x_{1,k} \in]0,1[$, $k = 1, 2, \ldots,$ $\lim_{k\to\infty} x_{1,k} = 0$, *such that*

$$\lim_{k\to\infty} \int_0^1 x_{1,k} \frac{\partial u_i(x_{1,k},x_2)}{\partial x_1} v(x_{1,k},x_2)\, dx_2 = 0,\ i = 1,2,3. \tag{14}$$

Proof For the proof of this lemma, see [11].

The following lemma play an essential role in the proof of unicity and existance of the weak solution the problem $(12)-(13)$. For the proof of this lemme, we use the density of the space $C_{0,\mathbf{\Gamma}}^1\left(\overline{\Omega}\right)$ in the space $W_{0,\mathbf{\Gamma}}^{1,2}(\Omega,\mu)$.

Lemma 2.2. *Let* $v \in W_{0,\mathbf{\Gamma}}^{1,2}(\Omega,\mu)$. *Then, we have the following apriori estimate*

$$\|v\|_{2,\mu} \le \frac{1}{2} |v|_{1,2,\mu}. \tag{15}$$

In addition, for all $\mathbf{v} = (v_1,v_2) \in \mathbb{V}$ *we have*

$$\left(\int_\Omega \left(\frac{v_1(x)}{x_1} \right)^2 d\mu(x) \right)^{1/2} \le \sqrt{2}\, |\mathbf{v}|_{1,2,\mu}. \tag{16}$$

Proof For all $v \in C_{0,\mathbf{\Gamma}}^1\left(\overline{\Omega}\right)$, we have

$$|v(x_1,x_2)| = \left| -\int_{x_1}^1 D_1 v(\xi_1,x_2)\, d\xi_1 \right| \le \left(\int_{x_1}^1 \frac{1}{\xi_1} d\xi_1 \right)^{\frac{1}{2}} \left(\int_0^1 x_1 |D_1 v(x)|^2 dx_1 \right)^{\frac{1}{2}},$$

from where, after multiplying by $x^{1/2}$, we derive

$$x_1 |v(x_1,x_2)|^2 \le (-x_1 \ln x_1) \int_0^1 x_1 |D_1 v(x)|^2 dx_1.$$

Integrating the above inequality over Ω, we obtain (because $-\int_0^1 x_1 \ln x_1 dx_1 = \frac{1}{4}$):

$$\|v\|_{2,\mu}^2 \le \frac{1}{4} \|D_1 v\|_{2,\mu}^2 \le \frac{1}{4} \left(\|D_1 v\|_{2,\mu}^2 + \|D_2 v\|_{2,\mu}^2 \right),$$

from where we deduce the estimate (15) by density of $C_{0,\mathbf{\Gamma}}^1\left(\overline{\Omega}\right)$ in $W_{0,\mathbf{\Gamma}}^{1,2}(\Omega,\mu)$.

For the proof of (16), let $\mathbf{v} = (v_1,v_2) \in \mathcal{V}$. Then, from the equality:

$$D_1(x_1 v_1) + x_1 D_1 v_1 = v_1 + x_1(D_1 v_1 + D_2 v_2) = 0.$$

We deduce

$$\frac{v_1}{x_1^{1/2}} = -x_1^{1/2}(D_1 v_1 + D_2 v_2).$$

Therefore, we obtain

$$\int_{\Omega}\left(\frac{v_1}{x_1}\right)^2 x_1dx \leq 2\int_{\Omega}\left((D_1v_1)^2+(D_2v_2)^2\right)x_1dx \leq 2\left(|v_1|^2_{1,2,\mu}+|v_2|^2_{1,2,\mu}\right)=2\,|v|^2_{1,2,\mu}$$

And by density of $\mathcal{V}$ in $\mathbb{V}$ we conclude (13).

The following theorem is a consequence of Theorem 2.2 in [10], see also [3].

Theorem 2.1. *Let* $p, q \in (1, +\infty)$ *such that*

$$\frac{1}{q}-\frac{1}{p}+\frac{1}{3}>0.$$

Then, for all $v \in W^{1,2}_{0,\mathbf{\Gamma}}(\Omega,\mu)$ *there is a constant* M_S *such that*

$$\|v\|_{q,\mu} \leq M_S\,|v|_{1,p,\mu}\,. \tag{17}$$

Let's rematk that, if $p = 2$ then the estimate (17) is true for all $1 < q < 6$.

Remark 2.1. The inequality in (15) is a Poincaré-type inequality. It can be obtained for a more general weight function, see ([6], p. 5).

So, the inequalities (15) and (17) justify the existence of the following constants:

$$P=\sup_{v\in W^{1,2}_{\mathbf{\Gamma}}(\Omega,\mu)}\frac{\|v\|_{2,\mu}}{|v|_{1,2,\mu}},\; S=\sup_{v\in W^{1,2}_{\mathbf{\Gamma}}(\Omega,\mu)}\frac{\|v\|_{4,\mu}}{|v|_{1,2,\mu}}, \tag{18}$$

called *Poincaré-Friedrichs's constant* and *Sobolev's constant*, respectively, and we have

$$P \leq \frac{1}{2} \text{ and } S \leq M_S. \tag{19}$$

3. Weak Formulation of the Problem

In this paragraph and those that follow, we will most often work in the space $\mathcal{V}$ and $\mathcal{W}$. The results obtained will be true in the spaces $\mathbb{V}$ and $\mathbb{W}$ by density of the first in the second.

First, let's go to the variational formulation of the problem $(12)-(13)$ in the space $\mathcal{W}$ by using the Lemma 2.1.

Let's put $\Omega_k =]x_{1,k}, 1[\times]0,1[$, where $x_{1,k} > 0$ is the point from the Lemma 2.1. Let $\mathbf{V} \in \mathcal{W}$ and multiply scalarly the equation (12) by $x_1\mathbf{V}=(x_1v_1, x_1v_2, x_1v_3)$ and integrate the result on Ω_k. We obtain the following integral equation:

$$\begin{aligned}-\textstyle\sum_{j=1}^{3}\nu_j\left(\int_{\Omega_k}D_1(x_1D_1u_j)\,v_jdx + +\displaystyle\int_{\Omega_k}D_2(x_1D_2u_j)\,v_jdx\right)+\nu\int_{\Omega_k}\frac{u_1v_1}{x_1^2}x_1dx\\+\sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\Omega_k}u_i(D_iu_j)\,v_jx_1dx=-\frac{1}{\rho_0}\int_{\Omega_k}\left(\frac{\partial p}{\partial x_1}v_1+\frac{\partial p}{\partial x_2}v_2\right)\\x_1dx+\beta_0\int_{\Omega_k}u_3v_1x_1dx+\int_{\Omega_k}gv_3x_1dx,\end{aligned} \tag{20}$$

$$\nu_1=\nu_2=\nu, \nu_3=\lambda.$$

Integrate by parts the terms $\int_{\Omega_k}D_i(x_1D_iu_j)\,v_jdx$,

$$i=1,2.$$

For $i = 1$, we obtain:

$$\int_{\Omega_k}D_1(x_1D_1u_j)\,v_jdx=\int_0^1 x_{1,k}D_1u_j(x_{1,k},x_2)\times v_j(x_{1,k},x_2)\,dx_2-\int_{\Omega_k}D_1u_j(x_1,x_2)\,D_1v_j(x_1,x_2)\,x_1dx.$$

Let's now pass to the limit in the above equality when $k \to \infty$, by using the Lemma 2.1. We derive

$$-\int_{\Omega}D_1(x_1D_1u_j)\,v_jdx=\int_{\Omega}D_1u_j(x_1,x_2)\,D_1v_j(x_1,x_2)\,x_1dx. \tag{21}$$

For $i = 2$, we obtain

$$\int_{\Omega_k} D_2 (x_1 D_2 u_j) v_j dx = \int_{x_{1,k}}^{1} x_1 D_2 u_j (x_1, x_2) v_j (x_1, x_2)|_{x_2=0}^{1} dx_1 - \int_{\Omega_k} D_2 u_j (x_1, x_2) D_2 v_j (x_1, x_2) x_1 dx$$

$$= - \int_{\Omega_k} D_2 u_j \times (x_1, x_2) D_2 v_j (x_1, x_2) x_1 dx,$$

because $v (x_1, 0) = v (x_1, 1) = 0, \forall x_1 \in (0, 1)$. Then, after passing to the limite when $k \to \infty$, we derive

$$- \int_{\Omega} D_2 (x_1 D_2 u_j) v_j dx = \int_{\Omega} D_2 u_j (x_1, x_2) D_2 v_j (x_1, x_2) x_1 dx. \tag{22}$$

On the other hand, because of the equality

$$\int_{\Omega} p \left(\frac{\partial (x_1 v_1)}{\partial x_1} + \frac{\partial (x_1 v_2)}{\partial x_2} \right) dx = \int_{\Omega} p \left(\frac{1}{x_1} \frac{\partial (x_1 v_1)}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) x_1 dx = 0,$$

We obtain, after passing to the limit $k \to \infty$

$$-\frac{1}{\rho_0} \int_{\Omega} \left(\frac{\partial p}{\partial x_1} v_1 + \frac{\partial p}{\partial x_2} v_2 \right) x_1 dx + \beta_0 \int_{\Omega} u_3 v_1 x_1 dx + \int_{\Omega_k} g v_3 x_1 dx = \beta_0 \int_{\Omega} u_3 v_1 x_1 dx + \int_{\Omega_k} g v_3 x_1 dx. \tag{23}$$

Therefore, by combining the equalities (21), (22) and (23) with the equality (20), we get

$$a_r(\mathbf{U}, \mathbf{V}) + b(\mathbf{u}, \mathbf{U}, \mathbf{V}) = (\mathbf{f} (\mathbf{U}) , \mathbf{V}) , \ \forall \mathbf{V} \in \mathcal{W}, \tag{24}$$

where

$$a_r(\mathbf{U}, \mathbf{V}) = \sum_{j=1}^{3} \nu_j \int_{\Omega} \left(\sum_{i=1}^{2} \sum_{j=1}^{3} \nu_j D_i u_j D_i v_j + \nu \frac{u_1 v_1}{x_1^2} \right) d\mu (x) , \tag{25}$$

$$b(\mathbf{u}, \mathbf{U}, \mathbf{V}) = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{U} \cdot \mathbf{V} d\mu (x) = \sum_{i=1}^{2} \sum_{j=1}^{3} \int_{\Omega} u_i (D_i u_j) v_j d\mu (x) , \tag{26}$$

$$(\mathbf{f} (\mathbf{U}) , \mathbf{V}) = \beta_0 \int_{\Omega} u_3 v_1 d\mu (x) + \int_{\Omega} g v_3 d\mu (x) . \tag{27}$$

Definition 3.1. *We say that a function* $\mathbf{U} \in \mathbb{W}$ *is a weak solution of the problem* $(12) - (13)$, *if* $\mathbf{U}$ *verify the integral equation* (24) *for all* $\mathbf{V} \in \mathbb{W}$.

Let's remark that $b(\cdot,\cdot, \cdot)$ is a trilinear form on the space $\mathbb{W}$.

Proposition 3.1. *For all* $\mathbf{U} = (\mathbf{u}, u_3) \in \mathbb{W}$ and $\mathbf{V}, \mathbf{W} \in \mathbb{W}$ *we have the following estimates:*

$$\nu_0 |\mathbf{U}|_{1,2,\mu}^2 \le a_r (\mathbf{U}, \mathbf{U}) \le 3\lambda_0 |\mathbf{U}|_{1,2,\mu}^2 , \tag{28}$$

$$|b (\mathbf{u}, \mathbf{V}, \mathbf{W})| \le \sqrt{2} S^2 |\mathbf{u}|_{1,2,\mu} |\mathbf{V}|_{1,2,\mu} |\mathbf{W}|_{1,2,\mu} \tag{29}$$

$$|(\mathbf{f} (\mathbf{U}) , \mathbf{V})| \le \left(\beta_0 P^2 |\mathbf{U}|_{1,2,\mu} + \gamma \right) |\mathbf{V}|_{1,2,\mu} , \tag{30}$$

where

$$\nu_0 = \min (\nu, \lambda) , \ \lambda_0 = \max (\nu, \lambda) , \ \gamma = \sup_{0 \ne v \in W_0^{1,2}(\Omega,\mu)} \frac{\left| \int_{\Omega} g v d\mu (x) \right|}{|v|_{1,2,\mu}}, \ g \in L_2 (\Omega, \mu) . \tag{31}$$

Proof Of (28). According to the definition of the form $a_r (\cdot, \cdot)$, we have

$$a_r(\mathbf{U}, \mathbf{U}) = \int_{\Omega} \left(\sum_{i=1}^{2} \sum_{j=1}^{3} \nu_j (D_i u_j)^2 + \frac{u_1^2}{x_1^2} \right) d\mu (x) \ge \nu_0 \sum_{i,j} \int_{\Omega} (D_i u_j)^2 d\mu (x) = \nu_0 \sum_{j=1}^{3} |u_j|_{1,2,\mu}^2 = \nu_0 |\mathbf{U}|_{1,2,\mu}^2 .$$

On the other hand, according the estimate (16), we obtain

$$a_r(\mathbf{U},\mathbf{U}) = \sum_{j=1}^{3} \nu_j |u_j|_{1,2,\mu}^2 + \nu \int_\Omega \frac{u_1^2}{x_1^2} d\mu \le \lambda_0 \left(|\mathbf{U}|_{1,2,\mu}^2 + 2|\mathbf{u}|_{1,2,\mu}^2 \right) \le 3\lambda_0 |\mathbf{U}|_{1,2,\mu}^2$$

Therefore, by combining the above inequality with the previous one, we obtain (28).
Proof of (29). Let's now apply the generalized Hölder's inequality to the second side of the following equality:

$$x_1 u_i D_i v_j w_j = \left(x_1^{1/4} u_i \right) \left(x_1^{1/2} D_i v_j \right) \left(x_1^{1/4} w_j \right).$$

We obtain

$$\left| \int_\Omega u_i D_i v_j w_j d\mu(x) \right| \le \left(\int_\Omega |u_i|^4 d\mu(x) \right)^{1/4} \times \left(\int_\Omega |D_i v_j|^2 d\mu(x) \right)^{1/2}$$

$$\times \left(\int_\Omega |w_j|^4 d\mu(x) \right)^{1/4} \le \|u_i\|_{4,\mu} |v_j|_{1,2,\mu} \|w_j\|_{4,\mu}.$$

According to the second equality of (18), we deduce:

$$\left| \int_\Omega u_i D_i v_j w_j d\mu(x) \right| \le S^2 |u_i|_{1,2,\mu} |v_j|_{1,2,\mu} |w_j|_{1,2,\mu}.$$

Therefore, according to the Cauchy-Schwarz inequality, we deduce from the inequality above

$$|b(\mathbf{u},\mathbf{V},\mathbf{W})| \le S^2 \sum_{i=1}^{2} |u_i|_{1,2,\mu} \left(\sum_{j=1}^{3} |v_j|_{1,2,\mu}^2 \right)^{1/2} \times \left(\sum_{j=1}^{3} |w_j|_{1,2,\mu}^2 \right)^{1/2}$$

$$= S^2 \sum_{i=1}^{2} |u_i|_{1,2,\mu} |\mathbf{V}|_{1,2,\mu} |\mathbf{W}|_{1,2,\mu} \le \sqrt{2} S^2 |\mathbf{u}|_{1,2,\mu} |\mathbf{V}|_{1,2,\mu} |\mathbf{W}|_{1,2,\mu}.$$

Proof of (30). Let $\mathbf{U} = (u_1, u_2, u_3) \in \mathbb{W}$ and $\mathbf{V} = (v_1, v_2, v_3) \in \mathbb{W}$. According to definition (27) and the inequality of Hölder taking and the first inequality in (18), we obtain

$$|(\mathbf{f}(\mathbf{U}), \mathbf{V})| \le \beta_0 \int_\Omega |u_3 v_1| d\mu + \left| \int_\Omega g(x) v_3 d\mu \right| \le \beta_0 \|u_3\|_{2,\mu} \|v_1\|_{2,\mu} + \sup_{v_3 \ne 0} \frac{\left| \int_\Omega g(x) v_3 d\mu(x) \right|}{|v_3|_{1,2,\mu}} |v_3|_{1,2,\mu}$$

$$\le \beta_0 P^2 |u_3|_{1,2,\mu} |v_1|_{1,2,\mu} + \gamma |v_3|_{1,2,\mu} \le \left(\beta_0 P^2 |\mathbf{U}|_{1,2,\mu} + \gamma \right) |\mathbf{V}|_{1,2,\mu}.$$

Lemma 3.1. *The bilinear form $a_r(\cdot,\cdot) : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$ is continuous, i.e. there is a constant $C = 3\lambda_0$* such that

$$|a_r(\mathbf{U},\mathbf{V})| \le 3\lambda_0 |\mathbf{U}|_{1,2} |\mathbf{V}|_{1,2}. \tag{32}$$

Proof Let $\mathbf{U} \in \mathbb{W}$ and $\mathbf{V} \in \mathbb{W}$. So, according to the inequalities of Hölder and Cauchy-Schwarz and the estimate (16), we obtain

$$|a_r(\mathbf{U},\mathbf{V})| = \left| \sum_{j=1}^{3} \sum_{i=1}^{2} \int_\Omega D_i u_j D_i v_j d\mu(x) + \nu \int_\Omega \frac{u_1 v_1}{x_1^2} d\mu(x) \right| \le$$

$$\le \lambda_0 \left(\sum_{j=1}^{3} \sum_{i=1}^{2} \|D_i u_j\|_{2,\mu} \|D_i v_j\|_{2,\mu} + 2|\mathbf{u}|_{1,2,\mu} |\mathbf{v}|_{1,2,\mu} \right) \le$$

$$\le \lambda_0 \left(\sum_{j=1}^{3} \left(\sum_{i=1}^{2} \|D_i u_j\|_{2,\mu}^2 \right)^{1/2} \left(\sum_{i=1}^{2} \|D_i v_j\|_{2,\mu}^2 \right)^{1/2} + +2|\mathbf{u}|_{1,2,\mu} |\mathbf{v}|_{1,2,\mu} \right) =$$

$$= \lambda_0 \left(\sum_{j=1}^{3} |u_j|_{1,2,\mu} |v_j|_{1,2,\mu} + 2|\mathbf{u}|_{1,2,\mu} |\mathbf{v}|_{1,2,\mu} \right) \le \lambda_0 \left(|\mathbf{U}|_{1,2} |\mathbf{V}|_{1,2} + 2|\mathbf{u}|_{1,2,\mu} |\mathbf{v}|_{1,2,\mu} \right) \le 3\lambda_0 |\mathbf{U}|_{1,2} |\mathbf{V}|_{1,2}.$$

Proposition 3.2. *Let* $\mathbf{u} \in \mathbb{V}$ and $\mathbf{V}, \mathbf{W} \in \mathbb{W}$. *Then*

$$b\left(\mathbf{u}, \mathbf{V}, \mathbf{V}\right) = 0, \tag{33}$$

$$b\left(\mathbf{u}, \mathbf{V}, \mathbf{W}\right) = -b\left(\mathbf{u}, \mathbf{W}, \mathbf{V}\right). \tag{34}$$

Proof Let $\mathbf{u} \in \mathcal{V}$ and $\mathbf{V} = (v_1, v_2, v_3) \in \mathcal{W}$. Then, using the integration by part, we obtain:

$$\sum_{i=1}^{2} \int_\Omega x_1 u_i \left(D_i v_j\right) v_j dx = \frac{1}{2}\left(\int_\Omega x_1 u_1 D_1 v_j^2 dx + \int_\Omega x_1 u_2 D_2 v_j^2 dx\right)$$

$$= \frac{1}{2}\left(\int_0^1 \left(x_1 u_1 v_j^2\big|_{x_1=0}^1\right) dx_2 - \int_\Omega D_1\left(x_1 u_1\right) v_j^2 dx\right) + \frac{1}{2}\left(\int_0^1 x_1 \left(u_2 v_j^2\left(x_1, x_2\right)\big|_{x_2=0}^1\right) dx_1 - \int_\Omega D_2\left(x_1 u_2\right) v_j^2 dx\right)$$

$$= -\frac{1}{2}\int_\Omega x_1 \left(\frac{1}{x_1} D_1\left(x_1 u_1\right) + D_2 u_2\right) v_j^2 dx = 0.$$

In the last equality above, we used the fact that $\left(x_1 u_1\right)\left(0, x_2\right) = 0$ (since $u_1 \in C^1\left(\overline{\Omega}\right)$) and $u_2\left(x_1, 0\right) = 0$ and the condition $\frac{1}{x_1} D_1\left(x_1 u_1\right) + D_2 u_2 = 0$. So, we get (33) by density of $\mathcal{V}$ (resp. $\mathcal{W}$) in $\mathbb{V}$ (resp. in $\mathbb{W}$).

From the property (33) we deduce (34). Indeed, according to the trilinearity of $b\left(\cdot,\cdot,\cdot\right)$, we obtain

$$0 = b\left(\mathbf{u}, \mathbf{V}+\mathbf{W}, \mathbf{V}+\mathbf{W}\right) = b(\mathbf{u}, \mathbf{V}, \mathbf{V}+\mathbf{W}) + b(\mathbf{u}, \mathbf{W}, \mathbf{V}+\mathbf{W})$$

$$= b(\mathbf{u}, \mathbf{V}, \mathbf{V}) + b(\mathbf{u}, \mathbf{V}, \mathbf{W}) + b(\mathbf{u}, \mathbf{W}, \mathbf{V}) + b(\mathbf{u}, \mathbf{W}, \mathbf{W}) = b(\mathbf{u}, \mathbf{V}, \mathbf{W}) + b(\mathbf{u}, \mathbf{W}, \mathbf{V}).$$

In the following, we shall establish the uniqueness and the exitence of the weak solution of the variational problem (24) in the space $\mathbb{W}$, by applying a similar approach to that followed in the case of the Navier-Stokes equations without coupling with the energy equation, see [6, 18].

4. Uniqueness of the Weak Solution

Lemma 4.1. Let the condition:

$$\nu_0 - \beta_0 P^2 > 0, \tag{35}$$

be fulfilled. *Then, if* $\mathbf{U} \in \mathbb{W}$ be a weak solution of (24), the estimate

$$|\mathbf{U}|_{1,2,\mu} \leq \frac{\gamma}{\nu_0 - \beta_0 P^2} \tag{36}$$

holds

Proof Let $\mathbf{U} \in \mathbb{W}$ be a weak solution of (24). Then, according the property (33) and the definition (27), we deduce

$$a_r\left(\mathbf{U}, \mathbf{U}\right) = \left(\mathbf{f}\left(\mathbf{U}\right), \mathbf{U}\right) = \beta_0 \int_\Omega u_3 u_1 d\mu + \int_\Omega g u_3 d\mu.$$

Therefore, by using the estimates (28) and (30),we obtain

$$\nu_0 \left|\mathbf{U}\right|_{1,2,\mu}^2 \leq a_r\left(\mathbf{U}, \mathbf{U}\right) = \left(f\left(\mathbf{U}\right), \mathbf{U}\right) \leq \beta_0 P^2 \left|\mathbf{U}\right|_{1,2,\mu}^2 + \gamma \left|\mathbf{U}\right|_{1,2,\mu}$$

From where, we deduce (36).

Proposition 4.1. *Let* $\mathbf{U} = \left(\mathbf{u}, u_3\right) \in \mathbb{W}$ *be a weak solution of the problem* (24) *and the following conditions are fulfilled*

$$\nu_0 - \beta_0 P^2 > 2^{1/4} \gamma^{1/2} S. \tag{37}$$

Then the weak solution $\mathbf{U}$ *is unique.*

Proof Let $\mathbf{U}^* = \left(\mathbf{u}^*, u_3^*\right) \in \mathbb{W}$ and $\mathbf{U}^{**} = \left(\mathbf{u}^{**}, u_3^{**}\right) \in \mathbb{W}$ be two solutions of the problem (24), i.e. $\forall \mathbf{V} \in \mathbb{W}$

$$a_r\left(\mathbf{U}^*, \mathbf{V}\right) + b\left(\mathbf{u}^*, \mathbf{U}^*, \mathbf{V}\right) = \left(\mathbf{f}\left(\mathbf{U}^*\right), \mathbf{V}\right),$$

$$a_r\left(\mathbf{U}^{**},\mathbf{V}\right)+b\left(\mathbf{u}^{**},\mathbf{U}^{**},\mathbf{V}\right)=\left(\mathbf{f}\left(\mathbf{U}^{**}\right),\mathbf{V}\right),$$

where $\mathbf{u}^*=(u_1^*,u_2^*)\in\mathbb{V}$ and $\mathbf{u}^{**}=(u_1^{**},u_2^{**})\in\mathbb{V}$.

Let's put $\overline{\mathbf{U}}=\mathbf{U}^*-\mathbf{U}^{**}$. Then,

$$a_r\left(\overline{\mathbf{U}},\mathbf{V}\right)+b\left(\mathbf{u}^*,\mathbf{U}^*,\mathbf{V}\right)-b\left(u^{**},\mathbf{U}^{**},\mathbf{V}\right)=\beta_0\int_\Omega\left(u_3^*-u_3^{**}\right)v_2d\mu. \tag{38}$$

On other hand,

$$b\left(\mathbf{u}^*,\mathbf{U}^*,\mathbf{V}\right)-b\left(\mathbf{u}^{**},\mathbf{U}^{**},\mathbf{V}\right)=b\left(\overline{\mathbf{u}},\mathbf{U}^*,\mathbf{V}\right)+b\left(\mathbf{u}^{**},\overline{\mathbf{U}},\mathbf{V}\right). \tag{39}$$

Let's put $\mathbf{V}=\overline{\mathbf{U}}$ in (38). Then, taking into account (39), we obtain the following equation

$$a_r\left(\overline{\mathbf{U}},\overline{\mathbf{U}}\right)+b\left(\overline{\mathbf{u}},\mathbf{U}^*,\overline{\mathbf{U}}\right)=\beta_0\int_\Omega\overline{u}_3\overline{u}_2d\mu\left(x\right), \tag{40}$$

According to the Hölder's inequality and the first equality in (18), we have

$$\left|\int_\Omega\overline{u}_3\overline{u}_2d\mu\left(x\right)\right|\le\left\|\overline{u}_3\right\|_{2,\mu}\left\|\overline{u}_2\right\|_{2,\mu}\le P^2\left|\overline{u}_3\right|_{1,2,\mu}\left|\overline{u}_2\right|_{1,2,\mu}. \tag{41}$$

So, according to the estimates (28), (29) and (30), we derive from (40) taking into account the estimates (16), (29) and (41):

$$\nu_0\left|\overline{\mathbf{U}}\right|_{1,2,\mu}^2\le a_r\left(\overline{\mathbf{U}},\overline{\mathbf{U}}\right)\le S^2\sqrt{6}\left|\overline{\mathbf{u}}\right|_{1,2,\mu}\left|\overline{\mathbf{U}}\right|_{1,2,\mu}\left|\mathbf{U}^*\right|_{1,2,\mu}+\beta_0P^2\left|\overline{u}_3\right|_{1,2,\mu}\left|\overline{u}_2\right|_{1,2,\mu}$$

$$\le\sqrt{2}S^2\left|\overline{\mathbf{U}}\right|_{1,2,\mu}^2\left|\mathbf{U}^*\right|_{1,2,\mu}+\beta_0P^2\left|\overline{\mathbf{U}}\right|_{1,2,\mu}^2\le\frac{\gamma S^2\sqrt{2}}{\nu_0-P^2\beta_0}\left|\overline{\mathbf{U}}\right|_{1,2,\mu}^2+\beta_0P^2\left|\overline{\mathbf{U}}\right|_{1,2,\mu}^2=\left(\frac{\gamma S^2\sqrt{2}}{\nu_0-P^2\beta_0}+\beta_0P^2\right)\left|\overline{\mathbf{U}}\right|_{1,2}^2$$

From where, we deduce

$$\left(\nu_0-\beta_0P^2-\frac{\sqrt{2}\gamma S^2}{\nu_0-P^2\beta_0}\right)\left|\overline{\mathbf{U}}\right|_{1,2,\mu}^2\le0.$$

If the conditions (34) are verified, then $\left|\overline{\mathbf{U}}\right|_{1,2,\mu}^2=0$, i.e. $\mathbf{U}^*=\mathbf{U}^{**}$.

5. Existence of the Weak Solution

We shall apply the Galerkin method to establish the existence of the weak solution of the variational problem (24). For the definition of this method see for example ([18], p. 134).

The space $\mathbb{W}$ is separable as a subspace of the space $\left(W_{0,\Gamma}^{1,2}\left(\Omega,\mu\right)\right)^3$ and it is a completion of $\mathcal{W}$ in $\left(W_{0,\Gamma}^{1,2}\left(\Omega,\mu\right)\right)^3$. Therefore, there is a sequence $\mathbf{W}_1$, $\mathbf{W}_2$, $\ldots$, $\mathbf{W}_m,\ldots$ of linearly independent elements of $\mathcal{W}$ which is total, i.e. it generates a dense subspace in $\mathbb{W}$. It is also linearly independent and total in $\mathcal{W}$.

For all fixed $m\ge1$, we define a sequence of an approximate solution $\mathbf{U}_m$ of the form

$$\mathbf{U}_m=\sum_{i=1}^m\xi_{i,m}\mathbf{W}_i,\ \xi_{i,m}\in\mathbb{R},\ i=1,\ldots,m, \tag{42}$$

such that

$$a_r(\mathbf{U}_m,\mathbf{W}_k)+b(\mathbf{u}_m,\mathbf{U}_m,\mathbf{W}_k)=\left(\mathbf{f}\left(\mathbf{U}_m\right),\mathbf{W}_k\right), \tag{43}$$
$$k=1,\ldots,m$$

The equations (43) define a system of nonlinear equations with respect to the unknowns $\xi_{i,m}\in\mathbb{R}$, $i=1,...,m$.

The existence and uniqueness of the solution of the system (43) follows from the following lemma.

Lemma 5.1. *Let X be a finite dimensional Hilbert space with scalar product $[\cdot,\cdot]$ and norm $[\cdot]$ and let $P:X\to X$ be a continuous operator such that*

$$\left[P\left(\eta\right),\eta\right]>0\,for\ \left[\eta\right]=r>0,\forall\eta\in X. \tag{44}$$

Then there is an element $\xi\in X$ *with* $[\xi]\le r$ *such that*

$$P\left(\xi\right)=0. \tag{45}$$

For the proof of the Lemma 5.2, see ([18], p. 134).

Let's apply the Lemma 5.1 to establish the existence of $\mathbf{U}_m$ in (42) verifying (43).

Let apply this lemma to prove the existence of $\mathbf{U}_m$ in the following way. As a space X of finite dimension, we take the space $\mathbb{W}_m$ generated by $\mathbf{W}_1$, $\mathbf{W}_2$, $\ldots$, $\mathbf{W}_m$. The scalar product in $\mathbb{W}_m$ is the induced scalar produc $a_r(\cdot,\cdot)$ of $\mathbb{W}$ and let $P_m:\mathbb{W}_m\to\mathbb{W}_m$ be the operator defined by the following equality:

$$a_r\left(P_m\left(\mathbf{U}\right),\mathbf{V}\right)=a_r\left(\mathbf{U},\mathbf{V}\right)+b\left(\mathbf{u},\mathbf{U},\mathbf{V}\right)-\left(\mathbf{f}\left(\mathbf{U}\right),\mathbf{V}\right), \tag{46}$$

$$V \in \mathbb{W}_m, \forall \mathbf{U} = (\mathbf{u}, u_3) \in \mathbb{W}_m, \forall \mathbf{V} \in \mathbb{W}_m.$$

Let us mention that the existence of such element $P_m(U) \in \mathbb{W}_m$ follows from Riesz's theorem, because the left side of (46) is actually a scalar product in $\mathbb{W}_m$ and the right hand side in (46) is a bounded linear functional of $V \in \mathbb{W}_m$.

Lemma 5.2. *The operator P_m defined by* (46) *is continuous on $\mathbb{W}_m$, i.e. for all sequence $\{\mathbf{U}_n\}_{n=1}^{\infty}$ in $\mathbb{W}_m$* convergent to $\mathbf{U}_0 \in \mathbb{W}_m$*:*

$$\lim_{n\to\infty} |\mathbf{U}_n - \mathbf{U}_0|_{1,2,\mu} = 0,$$

we have $\lim_{n\to\infty} P_m(\mathbf{U}_n) = P(\mathbf{U}_0)$ *in*$\mathbb{W}_m$.

Proof Let $(\mathbf{U}_n)$ be a sequence in $\mathbb{W}_m$ strongly convergent to $\mathbf{U}_0 \in \mathbb{W}_m$, i.e.

$$\lim_{n\to\infty} |\mathbf{U}_n - \mathbf{U}_0|_{1,2,\mu} = 0. \tag{47}$$

Then, because the trilinearity of $b(\cdot,\cdot,\cdot)$, for all $\mathbf{V} \in \mathbb{W}_m$, we have

$$\begin{aligned} a_r(P(\mathbf{U}_n) - P(\mathbf{U}_0), \mathbf{V}) &= a_r(\mathbf{U}_n - \mathbf{U}_0, \mathbf{V}) + b(\mathbf{u}_n, \mathbf{U}_n, \mathbf{V}) - b(\mathbf{u}_0, \mathbf{U}_0, \mathbf{V}) - (\mathbf{f}(\mathbf{U}_n), \mathbf{V}) + (\mathbf{f}(\mathbf{U}_0), \mathbf{V}) \\ &= a_r(\mathbf{U}_n - \mathbf{U}_0, \mathbf{V}) + b(\mathbf{u}_n - \mathbf{u}_0, \mathbf{U}_n, \mathbf{V}) + b(\mathbf{u}_0, \mathbf{U}_n - \mathbf{U}_0, \mathbf{V}) - (\mathbf{f}(\mathbf{U}_n) - \mathbf{f}(\mathbf{U}_0), \mathbf{V}). \end{aligned} \tag{48}$$

Let's now estimate the terms in the right side of (48).

According to the inequalities (31), (29), (30) and the limit (47), we obtain

$$|a_r(\mathbf{U}_n - \mathbf{U}_0, \mathbf{V})| \leq C |\mathbf{U}_n - \mathbf{U}_0|_{1,2,\mu} |\mathbf{V}|_{1,2,\mu} \to 0 \quad n \to \infty; \tag{49}$$

$$\begin{aligned} |b(\mathbf{u}_n - \mathbf{u}_0, \mathbf{U}_n, \mathbf{V}) + b(\mathbf{u}_0, \mathbf{U}_n - \mathbf{U}_0, \mathbf{V})| \leq C |\mathbf{V}|_{1,2,\mu} \Big(&|\mathbf{U}_n - \mathbf{U}_0|_{1,2,\mu} |U_n|_{1,2,\mu} \\ &+ |\mathbf{U}_n - \mathbf{U}_0|_{1,2,\mu} |\mathbf{u}_0|_{1,2,\mu} \Big) \to 0 \quad n \to \infty; \end{aligned} \tag{50}$$

$$|(\mathbf{f}(\mathbf{U}_n), \mathbf{V}) - (\mathbf{f}(\mathbf{U}_0), \mathbf{V})| \leq \beta_0 \left| \int_\Omega (u_{n,3} - u_{0,3}) v_2 d\mu(x) \right| \leq C |\mathbf{U}_n - \mathbf{U}_0|_{1,2,\mu} |\mathbf{V}|_{1,2,\mu} \to 0 \ \ n \to \infty. \tag{51}$$

Therefore, by combining the estimates $(49) - (51)$ with the equality (48), we derive

$$\begin{aligned} |a_r(P(\mathbf{U}_n) - P(\mathbf{U}_0), \mathbf{V})| &\leq |a(\mathbf{U}_n - \mathbf{U}_0, \mathbf{V})| + |b(\mathbf{u}_n - \mathbf{u}_0, \mathbf{U}_n, \mathbf{V})| \\ &+ |b(\mathbf{u}_0, \mathbf{U}_n - \mathbf{U}_0, \mathbf{V})| + |(\mathbf{f}(\mathbf{U}_n), \mathbf{V}) - (\mathbf{f}(\mathbf{U}_0), \mathbf{V})| \underset{n\to\infty}{\to} 0 \end{aligned}$$

i.e.

$$\lim_{n\to\infty} a_r(P(\mathbf{U}_n) - P(\mathbf{U}_0), \mathbf{V}) = 0, \ \forall \mathbf{V} \in \mathbb{W}_m.$$

Because the space $\mathbb{W}_m$ is finite, this is enough to establish the convergence $\lim_{n\to\infty} P(\mathbf{U}_n) = P(\mathbf{U}_0)$.

Now we will try to use Lemma 5.1. First, we are looking the condition (44). According to the apriori estimates (28) and (30), we obtain

$$\begin{aligned} [P_m(\mathbf{U}), \mathbf{U}] = a_r(P_m(\mathbf{U}), \mathbf{U}) &= a_r(\mathbf{U}, \mathbf{U}) - (\mathbf{f}(\mathbf{U}), \mathbf{U}) \geq \nu_0 |\mathbf{U}|^2_{1,2,\mu} - \left(\beta_0 P^2 |\mathbf{U}|_{1,2,\mu} + \gamma\right) |\mathbf{U}|_{1,2,\mu} \\ &= \left(\nu_0 - \beta_0 P^2\right) |\mathbf{U}|^2_{1,2,\mu} - \gamma |\mathbf{U}|_{1,2,\mu} = \left(\left(\nu_0 - \beta_0 P^2\right) |\mathbf{U}|_{1,2,\mu} - \gamma\right) |\mathbf{U}|_{1,2,\mu} \end{aligned}$$

i.e.

$$a_r(P_m(\mathbf{U}), \mathbf{U}) \geq \left(\left(\nu_0 - \beta_0 P^2\right) |\mathbf{U}|_{1,2,\mu} - \gamma\right) |\mathbf{U}|_{1,2,\mu}.$$

The above inequality implies that the condition (44) is fulfilled for $|\mathbf{U}|_{1,2,\mu} = r$, if

$$r > \frac{\gamma}{\nu_0 - \beta_0 P^2},$$

which is true for large enough r.

Thus, according to Lemma 5.1, we deduce that there exists a solution $\mathbf{U}_m$ of the form (42) verifying the system (43).

Let's now multiply (43) by $\xi_{k,m}$ and sum the obtained equalities for $k=1,\ldots,m$. We obtain

$$a_r\left(\mathbf{U}_m,\mathbf{U}_m\right)+b\left(\mathbf{u}_m,\mathbf{U}_m,\mathbf{U}_m\right)=\left(\mathbf{f}\left(\mathbf{U}_m\right),\mathbf{U}_m\right),$$

from where, in view of (32)

$$a_r\left(\mathbf{U}_m,\mathbf{U}_m\right)=\left(\mathbf{f}\left(\mathbf{U}_m\right),\mathbf{U}_m\right).$$

Therefore, using (28) and (30), we get

$$\left|\mathbf{U}_m\right|_{1,2,\mu}\leq\frac{\gamma}{\nu_0-\beta_0P^2}.$$

Since $\mathbf{U}_m$ is bounded sequence in $\mathbb{W}$, then there exists an element $\mathbf{U}\in\mathbb{W}$ and there is a subsequence $\mathbf{U}_{m'}$ such that

$$\mathbf{U}_{m'}\rightharpoonup\mathbf{U}\textit{ weakly in }\mathbb{W}\text{ and }\mathbf{U}_{m'}\to\mathbf{U}\textit{ in norm of }\quad\left(L_2\left(\Omega,\mu\right)\right)^3$$

i.e.

$$\lim_{m'\to\infty}a_r\left(\mathbf{U}_{m'},V\right)=a_r\left(\mathbf{U},\mathbf{V}\right),\ \forall\mathbf{V}\in\mathbb{W},\tag{52}$$

and

$$\lim_{m'\to\infty}\left\|\mathbf{U}_{m'}-\mathbf{U}\right\|_{2,\mu}=0.\tag{53}$$

Lemma 5.3. *If* $\left\{\mathbf{U}_n=\left(u_{1,n},u_{2,n},u_{3,n}\right)\right\}_{n=1}^{\infty}$ *be a sequence weakly converges to a function* $\mathbf{U}\in\mathbb{W}$, *strongly convergent in* $\left(L_2\left(\Omega,\mu\right)\right)^3$ *and uniformely bounded in* $\mathbb{W}$, *then*

$$\lim_{n\to\infty}b\left(\mathbf{u}_n,\mathbf{U}_n,\mathbf{V}\right)=b\left(\mathbf{u},\mathbf{U},\mathbf{V}\right),\ \forall\mathbf{V}\in\mathcal{W},\tag{54}$$

$$\lim_{n\to\infty}\left(\mathbf{f}\left(\mathbf{U}_n\right),\mathbf{V}\right)=\left(\mathbf{f}\left(\mathbf{U}\right),\mathbf{V}\right),\ \forall\mathbf{V}\in\mathcal{W}.\tag{55}$$

Proof Because the properties (33) and (34), we have

$$b\left(\mathbf{u}_n,\mathbf{U}_n,\mathbf{V}\right)=-b\left(\mathbf{u}_n,\mathbf{V},\mathbf{U}_n\right)=-\sum_{i,j}\int_\Omega u_{n,i}u_{n,j}D_iv_jd\mu$$

$$=-\sum_{i,j}\left(\int_\Omega\left(u_{n,i}-u_i\right)u_{n,j}D_iv_jd\mu+\ +\int_\Omega u_i\left(u_{n,j}-u_j\right)D_iv_jd\mu+\int_\Omega u_iu_jD_iv_jd\mu\right)$$

$$=-\sum_{i,j}\left(\int_\Omega\left(u_{n,i}-u_i\right)u_{n,j}D_iv_jd\mu+\ +\int_\Omega u_i\left(u_{n,j}-u_j\right)D_iv_jd\mu\right)+b\left(\mathbf{u},\mathbf{U},\mathbf{V}\right).\tag{56}$$

As $\mathbf{V}$ is smooth, $\mathbf{U}_n$ is uniformely bounded, then according to the Hölder inequality and the strongly convergence of $\left(\mathbf{U}_n\right)$ in $\left(L_2\left(\Omega,\mu\right)\right)^3$, we deduce

$$\left|\int_\Omega\left(u_{n,i}-u_i\right)u_{n,j}D_iv_jd\mu\right|\leq\left\|u_{n,i}-u_i\right\|_{2,\mu}\left\|u_{n,i}\right\|_{2,\mu}\left\|D_iv_j\left(x\right)\right\|_{\infty,0}\underset{n\to\infty}{\to}0.\tag{57}$$

Similarly, we have

$$\left|\int_\Omega u_i\left(u_{n,j}-u_j\right)D_iv_jd\mu\right|\leq\left\|u_{n,j}-u_j\right\|_{2,\mu}\left\|u_{n,i}\right\|_{2,\mu}\left\|D_iv_j\left(x\right)\right\|_{\infty,0}\underset{n\to\infty}{\to}0.\tag{58}$$

Therefore, by combining the inequalities (57) and (47) with the equality (56), we derive (54).

According to the Hölder's, we obtain

$$\left|\left(\mathbf{f}\left(\mathbf{U}_n\right),\mathbf{V}\right)-\left(\mathbf{f}\left(\mathbf{U}\right),\mathbf{V}\right)\right|\leq\beta_0\int_\Omega\left|u_{n,3}-u_3\right|\left|v_2\right|d\mu\leq\beta_0P^2\left|u_{n,3}-u_3\right|_{1,2,\mu}\left|v_3\right|_{1,2,\mu}\to0,\ \ n\to\infty,$$

i.e. we obtain (55)

Then we will go to the limit $m'\to\infty$ in (43) by the subsequence $\left(\mathbf{U}_{m'}\right)$ and by using the limits (52), (54) and (55) we deduce that for all $\mathbf{V}=\mathbf{W}_1,\mathbf{W}_2,\ldots$, we have

$$a_r\left(\mathbf{U},\mathbf{V}\right)+b\left(\mathbf{u},\mathbf{U},\mathbf{V}\right)=\left(f\left(\mathbf{U}\right),\mathbf{V}\right),\tag{59}$$

where $\mathbf{U}\in\mathbf{W}$ is a weak limit of the subsequence $\left(\mathbf{U}_{m'}\right)$. The equation (59) is also fulfilled for all linear combinaison $\mathbf{V}$ of $\mathbf{W}_1,\mathbf{W}_2,\ldots$ As these combinations are dences in $\mathbb{W}$, then the equation holds for all $\mathbf{V}\in\mathbb{W}$. In this way, we have proved the following theorem

Theorem 5.1. *Under the condition* (37), *the variational problem* (24) *admits a unique weak solution* $\mathbf{U}=\left(\mathbf{u},u_3\right)\in\mathbb{W}$.

6. Numerical Solution

In this paragraph, we will establish the existence and uniqueness of a numerical solution, obtained by the finite element method, approaching the weak solution of the variational problem (24).

6.1. Finite Element Method

Let $\mathcal{T}_h$ be a *regular triangulation* of $\overline{\Omega}$ formed by triangles $K_i \subset \overline{\Omega}$, $i = 1, \ldots, N$ of diameter h_{K_i}, *see* [7], and let put

$$h = \max_{K \in \mathcal{T}_h} h_K \to 0,$$

where $h_K = \max_{x,y \in K} |x - y|$.

Let $\mathcal{U}_h$ be the set of vertices of all the triangles $K \in \mathcal{T}_h$. We denote by $\overset{\circ}{\mathcal{U}}_{h,\Gamma}$ the set of vertices belonging to the interior of the domain $\Omega_\Gamma = \overline{\Omega} \diagdown \Gamma$. Here, the nodes located on the boundary $\Gamma_0 = \{(0, x_2) : 0 < x_2 < 1\}$ of the boundary $\partial\Omega$ belong to the set $\overset{\circ}{\mathcal{U}}_{h,\Gamma}$.

Here, any triangle $K \in \mathcal{T}_h$ has nonempty interior $\overset{\circ}{K}$ and Lipschit boundary ∂K, moreover, any two triangles K and K' satisfies the condition $\overset{\circ}{K} \cap \overset{\circ}{K'} \neq \emptyset$ and no vertex of one belongs to a side of the other.

Let $\mathcal{P}_1(K)$ be the space of polynomials of first degree defined on K. It is shown that there exist unique linearly independent polynomials $\lambda_{K,i}(x) \in P_1(K)$, $i = 1, 2, 3$, such that

$$\lambda_{K,i}(a_j) = \delta_{ij}, \ \sum_{i=1}^{3} \lambda_{K,i}(x) = 1. \tag{60}$$

called nodal functions or barycentric functions, where δ_{ij} is the Kronecker symbol, see

We call finite element the triple (K, P_K, Σ_K), where $\Sigma_K = \{\lambda_{K,i}\}_{i=1}^{3}$.

In the following instead of $\mathcal{P}_1(K)$ we simply write $\mathcal{P}_K$. In this way any polynomial $p \in \mathcal{P}_K$ can be written in the form

$$p(x) = \sum_{i=1}^{3} \lambda_{K,i}(x)\, p(a_{i,K}). \tag{61}$$

Note that given the geometric simplicity of our domain Ω, we have

$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}, \ \overline{K} = K \cup \partial K.$$

6.2. Definition of the Finite Element Approximation Space

At the triangulation $\mathcal{T}_h$ we associate the space $\mathcal{V}_h$ of functions $v_h \in C^1_{0,\Gamma}(\overline{\Omega})$ whose restriction $v_K = v_h|_K \in \mathcal{P}_K$, such that

$$\mathcal{V}_h = \left\{ v_h \in C^1_{0,\Gamma}(\overline{\Omega}) : v_{hK} = v_h|_K \in \mathcal{P}_K \right\}.$$

Note that the space $\mathcal{V}_h$ is a subspace of the weighted Sobolev space $W^{1,2}_{0,\Gamma}(\Omega, \mu) : \mathcal{V}_h \subset W^{1,2}_{0,\Gamma}(\Omega, \mu)$. In the following, we consider the space $\mathbb{V}_h$ of functions $\mathbf{v}_h \in \mathcal{V}_h \times \mathcal{V}_h$ such that

$$\int_K \mathrm{div}_r \mathbf{v}_h dx = 0 \ \forall K \in \mathcal{T}_h \tag{62}$$

and we put $\mathbb{W}_h = \mathbb{V}_h \times \mathcal{V}_h$ which is a subspace of $\left(W^{1,2}_{0,\Gamma}(\Omega, \mu) \right)^3$, which is called a *finite element approximation space*. The space $\mathbb{W}_h$ is a Hilbert space with respect of the induced scalar product:

$$((\mathbf{U}, \mathbf{V})) = \sum_{i=1}^{2} \sum_{j=1}^{3} \int_\Omega D_i u_j D_i v_j d\mu.$$

Let $\tau_B \subset \mathcal{T}_h$ be the set of all the elements of $\mathcal{T}_h$ having a common vertex $B \in \overset{\circ}{\mathcal{U}}_{h,\Gamma}$. Consider a continuous function w_{hB} on τ_B such that:

(i) its restriction $w_{hB}|_K = w_{hB,K}$ on each element $K \in \tau_B$ is a first degree polynomial $w_{hB,K} \in P_K$,

(ii) $w_{hB}(B) = 1$ and $w_{hB}(M) = 0 \ \forall M \in \overset{\circ}{\mathcal{U}}_{h,\Gamma}$, $M \neq B$.

According to the properties (60), we deduce that for all $\mathbf{U}_h = (\mathbf{u}_h, u_{h,3}) \in \mathbb{W}_h$ we have

$$\mathbf{U}_h = \sum_{B \in \overset{\circ}{\mathcal{U}}_{h,\Gamma}} \mathbf{U}_h(B)\, w_{hB} = \left(\sum_{B \in \overset{\circ}{\mathcal{U}}_{h,\Gamma}} \mathbf{u}_h(B)\, w_{hB}, \ \sum_{B \in \overset{\circ}{\mathcal{U}}_{h,\Gamma}} u_{h,3}(B)\, w_{hB} \right) \tag{63}$$

Note that the approximation space $\mathbb{W}_h$ is not a subspace of the energy space $\mathbb{W}$, $\mathbb{V}_h \subsetneq \mathbb{W}$, since for everything $\mathbf{v}_h(x) \in \mathbb{V}_h$ the divergence $\mathrm{div}_r \mathbf{v}_h(x)$ is not necessarily zero on Ω.

Proposition 6.1. For all $v_h \in \mathcal{V}_h$, there is a constant $P > 0$ no depending of h such that

$$\|v_h\|_{0,2,\mu} \leq P\, |v_h|_{1,2,\mu}, \tag{64}$$

where P is the Poincaré constant (see (18)):

$$P = \sup_{v_h \neq 0} \frac{\|v_h\|_{0,2,\mu}}{|v_h|_{1,2,\mu}} < 2.$$

.

Proof According to Green's formula, we have

$$\sum_{K\in\mathcal{T}_h}\int_K \frac{\partial}{\partial x_1}\left(x_1 v_K^2\right)dx = \sum_{K\in\mathcal{T}_h}\int_{\partial K} x_1 u_K^2 \nu_1 dx,$$

where $\nu_1 = \cos(x_1,\nu)$ is the exterior normal to ∂K. The scalar function $x_1^\alpha v_K^2$ is continuous on Ω, and therefore, on each side of the element $K\in\mathcal{T}_h$, moreover, it cancels out on Γ and on the boundary $\Gamma_0 = \{(0,x_2) : 0 < x_2 < 1\}$. In this way, if γ is a side common to two adjacent elements K and K' of $\mathcal{T}_h$, then we will have

$$x_1 v_K^2 \nu_1\big|_{\gamma\in\overline{K}} = -\, x_1 v_K^2 \nu_1\big|_{\gamma\in\overline{K'}}\,.$$

Therefore,

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} x_1 v_K^2 \nu_1 dx = 0.$$

On the other hand, we have

$$\int_\Omega \frac{\partial}{\partial x_1}\left(x_1 v_h^2\right)dx = \int_\Omega v_h^2 + 2x_1 v_h \frac{\partial v_h}{\partial x_1}dx = 0,$$

from where

$$\int_\Omega v_h^2 dx = -2\int_\Omega x_1 v_h \frac{\partial v_h}{\partial x_1}dx.$$

By Hölder's inequality, we get

$$\|v_h\|_{0,2}^2 = \int_\Omega v_h^2 dx \le 2\left(\int_\Omega x_1 v_h^2 dx\right)^{1/2}\left(\int_\Omega x_1\left(\frac{\partial v_h}{\partial x_1}\right)^2 dx\right)^{1/2} \le 2\,\|v_h\|_{0,2,\mu}\,|v_h|_{1,2,\mu}\,.$$

Therefore

$$\|v_h\|_{0,2,\mu}^2 = \int_\Omega x_1 v_h^2 dx \le \|v_h\|_{0,2}^2 \le 2\,\|v_h\|_{0,2,\mu}\,|v_h|_{1,2,\mu}\,,$$

i.e.

$$\|v_h\|_{0,2,\mu} \le 2\,|v_h|_{1,2,\mu}\,.$$

From where, we deduce

$$\frac{\|v_h\|_{0,2,\mu}}{|v_h|_{1,2,\mu}} \le P = \sup_{v_h\neq 0}\frac{\|v_h\|_{0,2,\mu}}{|v_h|_{1,2,\mu}} \le 2$$

The inequality (64) is similar to the Poincaré inequality.

Proposition 6.2. For all $v_h\in\mathcal{V}_h$, there is a constant $S>0$ no depending of h such that

$$\|v_h\|_{0,4,\mu} \le S\,|v_h|_{1,2,\mu}\,, \tag{65}$$

where $S\le P$ is the so-called Sobolev constant, see (18) .

Proof By Hölder's inequality, we get

$$\|v_h\|_{0,2,\mu}^4 = \int_\Omega \left(x_1^{1/4}|v_h|\right)^{1-\varepsilon}\left(x_1^{1/4}|v_h|\right)^{3+\varepsilon}dx \le \left(\int_\Omega x_1^{\frac{1}{2(1-\varepsilon)}}|v_h|^2 dx\right)^{(1-\varepsilon)/2}$$

$$\times\left(\int_\Omega \left(x_1^{1/4}|v_h|\right)^{2\frac{3+\varepsilon}{1+\varepsilon}}dx\right)^{(1+\varepsilon)/2} \le \left(\int_\Omega x_1^{\frac{1}{2(1-\varepsilon)}}|v_h|^2 dx\right)^{(1-\varepsilon)/2}$$

$$\times\left(\int_\Omega \left(x_1^{1/4}|v_h|\right)^{2\frac{3+\varepsilon}{1+\varepsilon}p}dx\right)^{(1+\varepsilon)/2p} \times \left(\mathrm{mes}\,(\Omega)\right)^{(1+\varepsilon)/2p'},$$

where $1/p+1/p'=1$. Therefore, taking $p = 2(1+\varepsilon)/(3+\varepsilon)$ in the above inequality, we get

$$\|v_h\|_{0,4,\mu}^4 \le \|v_h\|_{0,2,\mu}^{1-\varepsilon}\,\|v_h\|_{0,4,\mu}^{3+\varepsilon}\,,\ \mathrm{mes}\,(\Omega) = 1,$$

whence, taking into account (64) :

$$\|v_h\|_{0,4,\mu}^{1-\varepsilon} \leq \|v_h\|_{0,2,\mu}^{1-\varepsilon} \leq \left(P\,|v_h|_{1,2,\mu}\right)^{1-\varepsilon}.$$

Therefore

$$\frac{\|v_h\|_{0,4,\mu}}{|v_h|_{1,2,\mu}} \leq \sup_{v_h \neq 0} \frac{\|v_h\|_{0,4,\mu}}{|v_h|_{1,2,\mu}} = S \leq P$$

6.3. Formulation of the Discrete Problem

Let $\mathbf{u}_h \in \mathcal{V}_h \times \mathcal{V}_h$, $\mathbf{V}_h \in \mathbb{W}_h$, $\mathbf{W}_h \in \mathbb{W}_h$ and consider the following trilinear form:

$$b_h\left(\mathbf{u}_h, \mathbf{V}_h, \mathbf{W}_h\right) = b'\left(\mathbf{u}_h, \mathbf{V}_h, \mathbf{W}_h\right) + b''\left(\mathbf{u}_h, \mathbf{V}_h, \mathbf{W}_h\right), \tag{66}$$

where

$$b'\left(\mathbf{u}_h, \mathbf{V}_h, \mathbf{W}_h\right) = \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{2}\sum_{j=1}^{3} \frac{1}{2}\int_K u_{K,i}\left(D_i v_{K,j}\right) w_{K,j}\, d\mu,$$

$$b''\left(\mathbf{u}_h, \mathbf{V}_h, \mathbf{W}_h\right) = -\sum_{K\in\mathcal{T}_h} \sum_{i=1}^{2}\sum_{j=1}^{3} \frac{1}{2}\int_K u_{K,i} v_{K,j}\left(D_i w_{K,j}\right) d\mu.$$

It is easy to verify that for all $\mathbf{u}_h \in \mathcal{V}_h \times \mathcal{V}_h$ and $\mathbf{V}_h, \mathbf{W}_h \in \mathbb{W}_h$, we have

$$b_h\left(\mathbf{u}_h, \mathbf{V}_h, \mathbf{V}_h\right) = 0,\ b_h\left(\mathbf{u}_h, \mathbf{V}_h, \mathbf{W}_h\right) = -b_h\left(\mathbf{u}_h, \mathbf{W}_h, \mathbf{V}_h\right) \tag{67}$$

`Discreet problem`: we search $\mathbf{U}_h = (\mathbf{u}_h, u_{h,3}) \in \mathbb{W}_h$ of the form (63) such that

$$a_r\left(\mathbf{U}_h, \mathbf{V}_h\right) + b_h(\mathbf{u}_h, \mathbf{U}_h, \mathbf{V}_h) = \left(\mathbf{f}\left(u_{h,3}\right), \mathbf{V}_h\right),\ \forall \mathbf{V}_h \in \mathbb{W}_h \tag{68}$$

where $\mathbf{u}_h = (u_{h,1}, u_{h,2})$,

$$a_r(\mathbf{U}_h, \mathbf{V}_h) = \sum_{j=1}^{3}\sum_{i=1}^{2}\int_\Omega \nu_j (D_i u_{h,j})(D_i v_{h,j})d\mu + \nu\int_\Omega \frac{u_{h,1}v_{h,1}}{x_1^2}d\mu,$$

$$\left(\mathbf{f}\left(u_{h,3}\right), \mathbf{V}_h\right) = \beta_0\int_\Omega u_{h,3}v_{h,2}d\mu + \int_\Omega g v_{h,3}d\mu.$$

Proposition 6.3. Let $\mathbf{U}_h \in \mathbb{W}_h$ be a solution of (68) and let that the condition (35) holds. Then

$$|\mathbf{U}_h|_{1,2,\mu} \leq \frac{\gamma}{\nu_0 - \beta_0 P^2}, \tag{69}$$

where

$$\gamma = \sup_{0\neq v\in\mathcal{V}_h} \frac{\left|\int_\Omega g v d\mu\left(x\right)\right|}{|v|_{1,2,\mu}},\ g \in L_2\left(\Omega, \mu\right)$$

Proof Let $\mathbf{U}_h = (u_{h,1}, u_{h,2}, u_{h,3}) \in \mathbb{W}_h$ verify (65). Then, by property (70), we deduce that

$$a_r\left(\mathbf{U}_h, \mathbf{U}_h\right) = \left(\mathbf{f}\left(u_{h,3}\right), \mathbf{U}_h\right) = \beta_0\int_\Omega u_{h,3}u_{h,2}d\mu + \int_\Omega g u_{h,3}d\mu.$$

Hence, by Hölder's inequality, we get

$$\nu_0\left|\mathbf{U}_h\right|_{1,2,\mu}^2 \leq a_r\left(\mathbf{U}_h, \mathbf{U}_h\right) = \left(f\left(u_{h,3}\right), \mathbf{U}_h\right) \leq \beta_0\left\|\mathbf{U}_h\right\|_{0,2,\mu}^2 + \left|\int_\Omega g u_{h,3}d\mu\right|$$

$$\leq \beta_0\left\|\mathbf{U}_h\right\|_{0,2,\mu}^2 + \sup_{0\neq v_{h,3}\in\mathcal{V}_h}\frac{\left|\int_\Omega g v_{h,3}d\mu\left(x\right)\right|}{|v_{h,3}|_{1,2,\mu}}\left|u_{h,3}\right|_{1,2,\mu} \leq \beta_0 P^2\left|\mathbf{U}_h\right|_{1,2,\mu}^2 + \gamma\left|\mathbf{U}_h\right|_{1,2,\mu}.$$

From where, we deduce

$$\left(\nu_0 - \beta_0 P^2\right)\left|\mathbf{U}_h\right|_{1,2,\mu} \leq \gamma.$$

The inequality (69) is analogous to the inequality (36).
Let $\mathbf{U}_h = (\mathbf{u}_h, u_{h,3}) \in \mathbb{W}_h$ and $l_h(\mathbf{U}_h) : \mathbb{W}_h \to \mathbb{R}$ be a linear functional defined by the following equality:

$$\langle l_h(\mathbf{U}_h), \mathbf{V}_h\rangle = (f(u_{h,3}), \mathbf{V}_h). \tag{70}$$

The linearity here of $l_h(\mathbf{U}_h)$ is relative to $\mathbf{V}_h$.
Proposition 6.4. Let $\mathbf{U}_h = (\mathbf{u}_h, u_{h,3}) \in \mathbb{W}_h$ be a solution of (68). If the condition (35) holds, then

$$\|l_h(\mathbf{U}_h)\|^* = \sup_{\mathbf{V}_h \neq \mathbf{0}} \frac{|\langle l_h(\mathbf{U}_h), \mathbf{V}_h\rangle|}{|\mathbf{V}_h|_{1,2,\mu}} \leq \frac{\nu_0\gamma}{\nu_0 - \beta_0 P^2}. \tag{71}$$

Proof We have

$$|\langle l_h(\mathbf{U}_h), \mathbf{V}_h\rangle| = |(\mathbf{f}(u_{h,3}), \mathbf{V}_h)| \leq \beta_0 \|\mathbf{U}_h\|_{0,2,\mu} \|\mathbf{V}_h\|_{0,2,\mu} + \gamma |v_{h,3}|_{1,2,\mu} \leq \beta_0 P^2 |\mathbf{U}_h|_{1,2,\mu} |\mathbf{V}_h|_{1,2,\mu} + \gamma |\mathbf{V}_h|_{1,2,\mu}.$$

Then

$$\|l_h(\mathbf{U}_h)\|^* = \sup_{\mathbf{V}_h \neq \mathbf{0}} \frac{|| (\mathbf{f}(u_{h,3}), \mathbf{V}_h)||}{|\mathbf{V}_h|_{1,2,\mu}} \leq \beta_0 P^2 |\mathbf{U}_h|_{1,2,\mu} + \gamma \leq \frac{\gamma\beta_0 P^2}{\nu_0 - \beta_0 P^2} + \gamma = \frac{\nu_0\gamma}{\nu_0 - \beta_0 P^2}$$

Proposition 6.5. For all $\mathbf{u}_h \in (\mathcal{V}_h)^2 \subset \left(W^{1,2}_{0,\Gamma}(\Omega,\mu)\right)^2$ and $\mathbf{V}_h, \mathbf{W}_h \in (\mathcal{V}_h)^3 \subset \left(W^{1,2}_{0,\Gamma}(\Omega,\mu)\right)^3$, we have

$$|b_h(\mathbf{u}_h, \mathbf{V}_h, \mathbf{W}_h)| \leq \frac{S^2\sqrt{2}}{2} |\mathbf{u}_h|_{1,2,\mu} |\mathbf{V}_h|_{1,2,\mu} |\mathbf{W}_h|_{1,2,\mu}, \tag{72}$$

where S is the Sobolev constant, see (18).

Proof The estimate of the term $b'(\mathbf{u}_h, \mathbf{V}_h, \mathbf{W}_h)$. Let us apply Hölder's generalized inequality to the right side of the following equality:

$$\int_K u_{K,i}(D_i V_{K,j}) W_{K,j} d\mu = \int_K \left(x_1^{1/4} u_{K,i}\right)\left(x_1^{1/2} D_i v_{K,j}\right) \times \left(x_1^{1/4} w_{K,j}\right).$$

We find

$$\left|\int_K u_{K,i}(D_i v_{K,j}) w_{K,j} d\mu\right| \leq \left(\int_K |u_{K,i}|^4 d\mu(x)\right)^{1/4} \times$$

$$\times \left(\int_K |D_i v_{K,j}|^2 d\mu(x)\right)^{1/2} \times \left(\int_K |v_{K,j}|^4 d\mu(x)\right)^{1/4} \leq \|u_{K,i}\|_{0,4,\mu} |v_{K,j}|_{1,2,\mu} \|w_{K,j}\|_{0,4,\mu}.$$

According to (65), we deduce:

$$\left|\int_K u_{K,i}(D_i v_{K,j}) w_{K,j} d\mu\right| \leq S^2 |u_{K,i}|_{1,2,\mu} |v_{Kj}|_{1,2,\mu} |w_{K,j}|_{1,2,\mu}$$

where

$$|\varphi_K|_{1,2,\mu} = \left(\int_K \left(|D_1\varphi_K|^2 + |D_2\varphi_K|^2\right) d\mu\right)^{1/2}.$$

Therefore, applying the Cauchy-Schwarz inequality to the above inequality, we find

$$|b'(\mathbf{u}_h, \mathbf{V}_h, \mathbf{W}_h)| \leq \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{2} \sum_{j=1}^{3} \frac{1}{2} \left|\int_K u_{K,i}(D_i v_{K,j}) w_{K,j} d\mu\right| \leq$$

$$\leq \frac{S^2}{2} \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{2} |u_{K,i}|_{1,2,\mu} \left(\sum_{j=1}^{3} |v_{K,j}|^2_{1,2,\mu}\right)^{1/2} \times \left(\sum_{j=1}^{3} |w_{K,j}|^2_{1,2,\mu}\right)^{1/2}$$

$$\leq \tfrac{\sqrt{2}}{2} S^2 \textstyle\sum_{K\in\mathcal{T}_h} |\mathbf{u}_K|_{1,2,\mu} |\mathbf{V}_K|_{1,2,\mu} |\mathbf{W}_K|_{1,2,\mu} \leq \tfrac{S^2\sqrt{2}}{2} |\mathbf{u}_h|_{1,2,\mu} |\mathbf{V}_h|_{1,2,\mu} |\mathbf{W}_h|_{1,2,\mu}.$$

The estimate of the term $b''(\mathbf{u}_h, \mathbf{V}_h, \mathbf{W}_h)$. We proceed in the same way as in the previous case. We have

$$\left|\int_K u_{K,i} v_{K,j} \left(D_i w_{K,j}\right) d\mu\right| = \left|\int_K \left(x_1^{1/4} u_{K,i}\right)\left(x_1^{1/4} v_{K,j}\right)\left(x_1^{1/2} D_i w_{K,j}\right)\right|$$

$$\leq \left(\int_K |u_{K,i}|^4 d\mu(x)\right)^{1/4}\left(\int_K |v_{K,j}|^4 d\mu(x)\right)^{1/4} \times \left(\int_K |D_i w_{K,j}|^2 d\mu(x)\right)^{1/2}$$

$$\leq S^2 |u_{K,i}|_{1,2,\mu} |v_{K,j}|_{1,2,\mu} |w_{K,j}|_{1,2,\mu}.$$

Therefore, we get

$$|b''(\mathbf{u}_h, \mathbf{V}_h, \mathbf{W}_h)| \leq \frac{1}{2} \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{2} \sum_{j=1}^{3} \left|\int_K u_{K,i} v_{K,j} \left(D_i w_{K,j}\right) d\mu\right|$$

$$\leq \frac{S^2}{2} \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{2} |u_{K,i}|_{1,2,\mu} \left(\sum_{j=1}^{3} |v_{K,j}|_{1,2,\mu}^2\right)^{\frac{1}{2}} \times \left(\sum_{j=1}^{3} |w_{K,j}|_{1,2,\mu}^2\right)^{\frac{1}{2}}$$

$$\leq \frac{S^2\sqrt{2}}{2} |\mathbf{u}_h|_{1,2,\mu} |\mathbf{V}_h|_{1,2,\mu} |\mathbf{W}_h|_{1,2,\mu}$$

7. Existence and Uniqueness of the Approximate Solution

According to the Lemma 5.1, for all $h > 0$, there exists an approximate solution $\mathbf{U}_h = (\mathbf{u}_h, u_{h,3}) \in \mathbb{W}_h$ satisfying the discrete problem (68). Indeed, let us take $X = \mathbb{W}_h$ which is a finite dimensional Hilbert space, endowed with the induced scalar product $((\cdot,\cdot))$. Let $\mathcal{P} : \mathbb{W}_h \to \mathbb{W}_h$ be an application defined by

$$((\mathcal{P}(\mathbf{U}_h), \mathbf{V}_h)) = a_r(\mathbf{U}_h, \mathbf{V}_h) + b_h(\mathbf{u}_h, \mathbf{U}_h, \mathbf{V}_h) - \langle l(\mathbf{U}_h), \mathbf{V}_h\rangle,$$

for all $\mathbf{U}_h = (\mathbf{u}_h, u_{h,3}) \in \mathbb{W}_h$ and $\mathbf{V}_h \in \mathbb{W}_h$.

From (69), (71) and (72), we show that the operator $\mathcal{P}$ is continuous. On the other hand, from (68), (69) and (71), we have

$$((\mathcal{P}(\mathbf{U}_h), \mathbf{U}_h)) = a_r(\mathbf{U}_h, \mathbf{U}_h) - \langle l(\mathbf{U}_h), \mathbf{U}_h\rangle \geq |\mathbf{U}_h|_{1,2,\mu}\left(\nu_0 |\mathbf{U}_h|_{1,2,\mu} - \frac{\nu_0\gamma}{\nu_0 - \beta_0 P^2}\right).$$

Therefore, if $|\mathbf{U}_h|_{1,2,\mu} = k$ and $k > \dfrac{\gamma}{\nu_0 - \beta_0 P^2}$ then $((\mathcal{P}(\mathbf{U}_h), \mathbf{U}_h)) > 0$. In this way, the condition (45) holds, and so there exists at least one element $\mathbf{U}_h \in \mathbb{W}_h$ such that

$$((\mathcal{P}(\mathbf{U}_h), \mathbf{V}_h)) = 0 \ \ \forall \mathbf{V}_h \in \mathbb{W}_h,$$

which is equivalent to the equation (68).

Suppose there are two solutions $\mathbf{U}_h^*$ and $\mathbf{U}_h^{**}$ of the equation (68). Let's pose $\mathbf{U}_h = \mathbf{U}_h^* - \mathbf{U}_h^{**} = (\mathbf{u}_h, u_{h,3})$. Then

$$a_r(\mathbf{U}_h, \mathbf{V}_h) + b_h(\mathbf{u}_h^*, \mathbf{U}_h^*, \mathbf{V}_h) - b_h(\mathbf{u}_h^{**}, \mathbf{U}_h^{**}, \mathbf{V}_h) = \beta_0 \int_\Omega u_{h,3} v_{h,2} d\mu.$$

On the other hand, we have

$$b_h(\mathbf{u}_h^*, \mathbf{U}_h^*, \mathbf{V}_h) - b(\mathbf{u}_h^{**}, \mathbf{U}_h^{**}, \mathbf{V}_h) = b(\mathbf{u}_h, \mathbf{U}_h^*, \mathbf{V}_h) + b(\mathbf{u}_h^{**}, \mathbf{U}_h, \mathbf{V}_h). \quad (73)$$

Let's take $\mathbf{V}_h = \mathbf{U}_h$ in (73). Then, taking into account (67), we get

$$a_r(\mathbf{U}_h, \mathbf{U}_h) + b(\mathbf{u}_h, \mathbf{U}_h^*, \mathbf{U}_h) = \beta_0 \int_\Omega u_{h,3} u_{h,2} d\mu(x). \quad (74)$$

By Hölder's inequality and (64), we find

$$\left| \int_\Omega u_{h,3} u_{h,2} d\mu(x) \right| \leq \|u_{h,3}\|_{2,\mu} \|u_{h,2}\|_{2,\mu} \leq P^2 |u_{h,3}|_{1,2,\mu} |u_{h,2}|_{1,2,\mu} . \quad (75)$$

On the other hand, by (72), we have

$$|b_h(\mathbf{u}_h, \mathbf{U}_h^*, \mathbf{U}_h)| \leq \frac{S^2\sqrt{2}}{2} |\mathbf{u}_h|_{1,2,\mu} |\mathbf{U}_h|_{1,2,\mu} |\mathbf{U}_h^*|_{1,2,\mu} \quad (76)$$

Thus, by combining (75) and (76) with (74) and taking into account (69), we obtain

$$\nu_0 |\mathbf{U}_h|^2_{1,2,\mu} \leq a_r(\mathbf{U}_h, \mathbf{U}_h) \leq |b(\mathbf{u}_h, \mathbf{U}_h^*, \mathbf{U}_h)| + \beta_0 \left| \int_\Omega u_{h,3} u_{h,2} d\mu(x) \right| \leq$$

$$\leq \frac{S^2\sqrt{2}}{2} |\mathbf{u}_h|_{1,2,\mu} |\mathbf{U}_h|_{1,2,\mu} |\mathbf{U}_h^*|_{1,2,\mu} + \beta_0 P^2 |u_{h,3}|_{1,2,\mu} |u_{h,2}|_{1,2,\mu}$$

$$\leq \left(\frac{S^2\sqrt{2}}{2} |\mathbf{U}_h^*|_{1,2,\mu} + \beta_0 P^2 \right) |\mathbf{U}_h|^2_{1,2,\mu} \leq \left(\frac{\gamma S^2 \sqrt{2}}{2(\nu_0 - \beta_0 P^2)} + \beta_0 P^2 \right) |\mathbf{U}_h|^2_{1,2,\mu} ,$$

From where, we derive

$$\left(\nu_0 - \beta_0 P^2 - \frac{\gamma S^2 \sqrt{2}}{2(\nu_0 - \beta_0 P^2)} \right) |\mathbf{U}_h|^2_{1,2,\mu} \leq 0.$$

Thus, if the following inequality:

$$\left(\nu_0 - \beta_0 P^2\right)^2 - \frac{\sqrt{2}}{2} \gamma S^2 > 0, \ \nu_0 - \beta_0 P^2 > 0 \quad (77)$$

holds, then $|\mathbf{U}_h|_{1,2,\mu} = 0$, i.e. $\mathbf{U}_h^* = \mathbf{U}_h^{**}$.

In this way, we have just proved the existence and uniqueness theorem of the following approximate solution:

Theorem 7.1. If the conditions (77) hold, then for all $h > 0$, there exists a single element $\mathbf{U}_h \in \mathbb{W}_h$ solution of the discrete problem (68).

8. Conclusion

The main objective of this work is the mathematical study of the stationary flow of an incompressible Newtonian fluid governed by the Navier-Stokes equations in a cylinder. It is a question of establishing the existence and the unicity of the weak and numerical solution of the variational problem associated to this problem with respect to the cylindrical coordinates by supposing that the data of the problem do not depend on the angle of rotation. In this case, we are reduced to dealing with the problem in dimension two instead of three. However, the transition from Cartesian coordinates to cylindrical coordinates gives rise to a problem which degenerates on the axis of symmetry. Its study therefore requires the introduction of an appropriate functional space, namely the weighted Sobolev space. On the other hand, this study allows us to establish the necessary algebraic conditions which must be verified by the values of the physical parameters of the problem, see (37) and (77).

In this work, we do not deal with the question of numerical simulation which is the subject of another work in preparation to be published later.

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