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The Orthogonality of Two-scale Three-dimensional Eight-direction Wavelet

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Abstract: The construction of wavelets is a key problem in wavelet analysis. In the background of the one-dimensional double wavelet theory and the one dimensional biorthogonal bidirectional wavelet construction theory, this paper extends the one-dimensional bidirectional wavelet to the two-scale three-dimensional eight-direction biorthogonal wavelet. By using the method of tensor products to construct higher dimensional wavelets, the two-scale three-dimensional eight-direction multi-resolution analysis, two-scale three-dimensional eight-direction scale function and wavelet function are obtained. In addition, the conditions satisfied of the orthogonal and biorthogonal properties of the two-scale three-dimensional eight-direction wavelet are studied.

Keywords: Two-scale Three-dimensional Eight-direction Wavelet, Two-scale Three-dimensional Eight-direction Multiresolution Analysis, Orthogonal Wavelet

1. Introduction

The core problem of wavelet theory is to study of the structure and properties of wavelet. In order to make up for the deficiency of single wavelet, people put forward multi-wavelet theory, such as Gabor wavelet [1]. L. Shen et al proposed the symmetry-antisymmetric orthonormal multiwavelets [2], J Lebrun and M Vetterli studied the regularity problem of multiwavelets [3]. Yang Shouzhi et al gave the method of the constructing compactly supported orthonormal multiscale functions [4-8], and J Lian et al studied the orthogonal multiwavelet problem [9-12]. Professor Yang proposed the concepts of bidirectional subdivision equation and bidirectional wavelet, and discussed the orthogonality, approximation order and regularity of bidirectional subdivision function [13-14]. Especially, the bidirectional addition function is the extension of the single scale function. Based on the construction algorithm of a-scale three-dimensional eight-direction plus fine wave scale function [15] and biorthogonal a-scale three-dimensional eight-direction scale function and wavelet function [16], this paper extended the one-dimensional bidirectional wavelet to the two-scale three-dimensional eight-direction biorthogonal wavelet by tensor product. Furthermore, the two-scale three-dimensional eight-direction multi-resolution analysis is given, the construction of two-scale three-dimensional eight-direction scale function and wavelet function is studied, and the two-scale three-dimensional eight-directionl wavelet under orthogonal and biorthogonal conditions is obtained.

Introduction mark: $\forall g(x_1,x_2,x_3), f(x_1,x_2,x_3)\in L^2(R^3)$, and define the inner product as follows:

$$\langle g(x_1,x_2,x_3), f(x_1,x_2,x_3)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1,x_2,x_3)\overline{f(x_1,x_2,x_3)}dx_1dx_2dx_3 \tag{1}$$

Where $\overline{f(x_1,x_2,x_3)}$ is the complex conjugate of $f(x_1,x_2,x_3)$. The Fourier transform of $g(x_1,x_2,x_3)$ is defined as:

$$\hat{g}(\omega_1,\omega_2,\omega_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(x_1\omega_1+x_2\omega_2+x_3\omega_3)} g(x_1,x_2,x_3)dx_1dx_2dx_3 \tag{2}$$

The tensor product is defined as follows: let G, F and J are the three unary function spaces, the basis of G is $\{g_k(x)\}_{k\in z}$, the basis of F is $\{f_k(y)\}_{k\in z}$, and the basis of J is $\{j_k(z)\}_{k\in Z}$, therefore, the tensor product space of G, F and J is a ternary function space with $\{g_k(x)f_k(y)j_k(z)\}_{k\in Z}$ as the basis, namely, $g\otimes f\otimes j$; Thus: $\varphi(x,y,z)=\varphi(x)\varphi(y)\varphi(z)$ is true for a function of three variables.

2. Two-Scale Three-dimensional Eight-direction Multi-resolution Analysis

Definition 2.1. Let the scale function of two-scale three-dimensional eight-direction wavelet be set $\varphi(x_1,x_2,x_3)\in L^2(R^3)$, define the subspace sequence $\{V_j\}_{j\in Z}\subset L^2(R^3)$:

$$\begin{aligned} V_j = Clos_{L^2(R^3)} < & 2^j\varphi(2^j x_1-k_1,2^j x_2-k_2,2^j x_3-k_3), \\ & 2^j\varphi(k_1-2^j x_1,2^j x_2-k_2,2^j x_3-k_3), \\ & 2^j\varphi(2^j x_1-k_1,k_2-2^j x_2,2^j x_3-k_3), \\ & 2^j\varphi(2^j x_1-k_1,2^j x_2-k_2,k_3-2^j x_3), \\ & 2^j\varphi(k_1-2^j x_1,k_2-2^j x_2,2^j x_3-k_3), \\ & 2^j\varphi(k_1-2^j x_1,2^j x_2-k_2,k_3-2^j x_3), \\ & 2^j\varphi(2^j x_1-k_1,k_2-2^j x_2,k_3-2^j x_3), \\ & 2^j\varphi(k_1-2^j x_1,k_2-2^j x_2,k_3-2^j x_3):k_1,k_2,k_3\in Z > \end{aligned} \tag{3}$$

Then a multi-resolution analysis in $L^2(R^3)$ is generated with $(MRA)\{V_j\}_{j\in Z}$ has and only has $\{V_j\}_{j\in Z}$, defined by Equation (3) satisfies:

(i) $\cdots\subset V_{-1}\subset V_0\subset V_2\subset\cdots$;

(ii) $Clos_{L^2(R^3)}\bigcup\limits_{j\in Z}V_j=L^2(R^3)$;

(iii) $\bigcap\limits_{j\in Z}V_j=\{0\}$;

(iV) $f(x_1,x_2,x_3)\in V_j\Leftrightarrow f(2x_1,2x_2,2x_3)\in V_{j+1}$;

(V) There exists a scale function $\varphi(x_1,x_2,x_3)$ in $L^2(R^3)$, make the set of functions

$$\begin{aligned} \{ & \varphi(x_1-k_1,x_2-k_2,x_3-k_3), \\ & \varphi(k_1-x_1,x_2-k_2,x_3-k_3), \\ & \varphi(x_1-k_1,k_2-x_2,x_3-k_3), \\ & \varphi(x_1-k_1,x_2-k_2,k_3-x_3), \\ & \varphi(k_1-x_1,k_2-x_2,x_3-k_3), \\ & \varphi(k_1-x_1,x_2-k_2,k_3-x_3), \\ & \varphi(x_1-k_1,k_2-x_2,k_3-x_3), \\ & \varphi(k_1-x_1,k_2-x_2,k_3-x_3):k_1,k_2,k_3\in Z\} \end{aligned}$$

It's a Riesz basis of V_0. Hence, two constants $0<C_1\le C_2<\infty$, can be found:

$$C_1\left\|\sum_{k_1,k_2,k_3\in Z}c_{k_1,k_2,k_3}\overline{c_{k_1,k_2,k_3}}\right\|_2^2 \le \left\|\begin{aligned} & \sum_{k_1,k_2,k_3\in Z}c^{+,+,+}_{k_1,k_2,k_3}\varphi(x_1-k_1,x_2-k_2,x_3-k_3) \\ & +\sum_{k_1,k_2,k_3\in Z}c^{+,+,-}_{k_1,k_2,k_3}\varphi(x_1-k_1,x_2-k_2,k_3-x_3) \\ & +\sum_{k_1,k_2,k_3\in Z}c^{+,-,+}_{k_1,k_2,k_3}\varphi(x_1-k_1,k_2-x_2,x_3-k_3) \\ & +\sum_{k_1,k_2,k_3\in Z}c^{+,-,-}_{k_1,k_2,k_3}\varphi(x_1-k_1,k_2-x_2,k_3-x_3) \\ & +\sum_{k_1,k_2,k_3\in Z}c^{-,+,+}_{k_1,k_2,k_3}\varphi(k_1-x_1,x_2-k_2,x_3-k_3) \\ & +\sum_{k_1,k_2,k_3\in Z}c^{-,+,-}_{k_1,k_2,k_3}\varphi(k_1-x_1,x_2-k_2,k_3-x_3) \\ & +\sum_{k_1,k_2,k_3\in Z}c^{-,-,+}_{k_1,k_2,k_3}\varphi(k_1-x_1,k_2-x_2,x_3-k_3) \\ & +\sum_{k_1,k_2,k_3\in Z}c^{-,-,-}_{k_1,k_2,k_3}\varphi(k_1-x_1,k_2-x_2,k_3-x_3) \end{aligned}\right\|_2^2 \le C_2\left\|\sum_{k_1,k_2,k_3\in Z}c_{k_1,k_2,k_3}\overline{c_{k_1,k_2,k_3}}\right\|_2^2 \tag{4}$$

Where

$$\{c_{k_1,k_2,k_3}\}=\left\{\begin{aligned} & c^{+,+,+}_{k_1,k_2,k_3},c^{+,+,-}_{k_1,k_2,k_3},c^{+,-,+}_{k_1,k_2,k_3},c^{+,-,-}_{k_1,k_2,k_3}, \\ & c^{-,+,+}_{k_1,k_2,k_3},c^{-,+,-}_{k_1,k_2,k_3},c^{-,-,+}_{k_1,k_2,k_3},c^{-,-,-}_{k_1,k_2,k_3} \end{aligned}\right\}_{k_1,k_2,k_3\in Z}$$

is an any sequence in $l^2(Z^3)$, and equation (4) can be called a stable condition.

Definition 2.2. The tensor product of three unitary bidirectional simple wavelets $\varphi(x),\varphi(y),\varphi(z)$ is used to construct three-dimensional eight-direction bidirectional wavelets in three-dimensional space.

Let the bidirectional simple wavelet subdivision function $\varphi(x),\varphi(y),\varphi(z)$ satisfying the subdivision equation as follows:

$$\varphi(x)=\sum_{k\in Z}p_{1,k}^{+}\varphi(2x-k)+\sum_{k\in Z}p_{1,k}^{-}\varphi(k-2x) \tag{5}$$

$$\varphi(y)=\sum_{l\in Z}p_{2,l}^{+}\varphi(2y-l)+\sum_{l\in Z}p_{2,l}^{-}\varphi(l-2y) \tag{6}$$

$$\varphi(z)=\sum_{t\in Z}p_{3,t}^{+}\varphi(2z-t)+\sum_{t\in Z}p_{3,t}^{-}\varphi(t-2z) \tag{7}$$

It can be obtained from $\varphi(x,y,z)=\varphi(x)\varphi(y)\varphi(z)$:

$$\begin{aligned}
\varphi(x,y,z)&=\left\{\sum_{k\in Z}p_{1,k}^{+}\varphi(2x-k)+\sum_{k\in Z}p_{1,k}^{-}\varphi(k-2x)\right\}\otimes\left\{\sum_{l\in Z}p_{2,l}^{+}\varphi(2y-l)+\sum_{l\in Z}p_{2,l}^{-}\varphi(l-2y)\right\}\otimes\left\{\sum_{t\in Z}p_{3,t}^{+}\varphi(2z-t)+\sum_{t\in Z}p_{3,t}^{-}\varphi(t-2z)\right\}\\
&=\sum_{k,l,t\in Z}p_{1,k}^{+}p_{2,l}^{+}p_{3,t}^{+}\varphi(2x-k,2y-l,2z-t)+\sum_{k,l,t\in Z}p_{1,k}^{+}p_{2,l}^{+}p_{3,t}^{-}\varphi(2x-k,2y-l,t-2z)\\
&+\sum_{k,l,t\in Z}p_{1,k}^{+}p_{2,l}^{-}p_{3,t}^{+}\varphi(2x-k,l-2y,2z-t)+\sum_{k,l,t\in Z}p_{1,k}^{+}p_{2,l}^{-}p_{3,t}^{-}\varphi(2x-k,l-2y,t-2z)\\
&+\sum_{k,l,t\in Z}p_{1,k}^{-}p_{2,l}^{+}p_{3,t}^{+}\varphi(k-2x,2y-l,2z-t)+\sum_{k,l,t\in Z}p_{1,k}^{-}p_{2,l}^{+}p_{3,t}^{-}\varphi(k-2x,2y-l,t-2z)\\
&+\sum_{k,l,t\in Z}p_{1,k}^{-}p_{2,l}^{-}p_{3,t}^{+}\varphi(k-2x,l-2y,2z-t)+\sum_{k,l,t\in Z}p_{1,k}^{-}p_{2,l}^{-}p_{3,t}^{-}\varphi(k-2x,l-2y,t-2z)
\end{aligned} \tag{8}$$

Select the corresponding sequence from above to get:

$$\left\{p_{k_1,k_2,k_3}^{+,+,+}\right\}_{k_1,k_2,k_3\in Z},\left\{p_{k_1,k_2,k_3}^{+,+,-}\right\}_{k_1,k_2,k_3\in Z},\left\{p_{k_1,k_2,k_3}^{+,-,+}\right\}_{k_1,k_2,k_3\in Z},\left\{p_{k_1,k_2,k_3}^{+,-,-}\right\}_{k_1,k_2,k_3\in Z},$$
$$\left\{p_{k_1,k_2,k_3}^{-,+,+}\right\}_{k_1,k_2,k_3\in Z},\left\{p_{k_1,k_2,k_3}^{-,+,-}\right\}_{k_1,k_2,k_3\in Z},\left\{p_{k_1,k_2,k_3}^{-,-,+}\right\}_{k_1,k_2,k_3\in Z},\left\{p_{k_1,k_2,k_3}^{-,-,-}\right\}_{k_1,k_2,k_3\in Z}$$

Then we get:

$$\begin{aligned}
\varphi(x_1,x_2,x_3)&=\sum_{k_1,k_2,k_3\in Z}p_{k_1,k_2,k_3}^{+,+,+}\varphi(2x_1-k_1,2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}p_{k_1,k_2,k_3}^{+,+,-}\varphi(2x_1-k_1,2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}p_{k_1,k_2,k_3}^{+,-,+}\varphi(2x_1-k_1,k_2-2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}p_{k_1,k_2,k_3}^{+,-,-}\varphi(2x_1-k_1,k_2-2x_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}p_{k_1,k_2,k_3}^{-,+,+}\varphi(k_1-2x_1,2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}p_{k_1,k_2,k_3}^{-,+,-}\varphi(k_1-2x_1,2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}p_{k_1,k_2,k_3}^{-,-,+}\varphi(k_1-2x_1,k_2-2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}p_{k_1,k_2,k_3}^{-,-,-}\varphi(k_1-2x_1,k_2-2x_2,k_3-2x_3)
\end{aligned} \tag{9}$$

If equation (9) has a good solution, then the result of three-dimensional wavelet is an example of a two-scale three-dimensional eight-direction subdivision function.

If equation (9) can be reduced to a three-dimensional subdivision function, it is required that $p_{k_1,k_2,k_3}^{+,+,-}$, $p_{k_1,k_2,k_3}^{+,-,+}$, $p_{k_1,k_2,k_3}^{+,-,-}$, $p_{k_1,k_2,k_3}^{-,+,+}$, $p_{k_1,k_2,k_3}^{-,+,-}$, $p_{k_1,k_2,k_3}^{-,-,+}$, $p_{k_1,k_2,k_3}^{-,-,-}$ are 0. Let the corresponding three-dimensional eight-direction two-scale function in $\left\{V_j\right\}_{j\in Z}$ is $\varphi(x_1,x_2,x_3)$, the two-scale function of the corresponding tight support is $\tilde{\varphi}(x_1,x_2,x_3)$, then:

$$\begin{aligned}
\tilde{\varphi}(x_1,x_2,x_3)&=\sum_{k_1,k_2,k_3\in Z}\tilde{p}_{k_1,k_2,k_3}^{+,+,+}\tilde{\varphi}(2x_1-k_1,2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}\tilde{p}_{k_1,k_2,k_3}^{+,+,-}\tilde{\varphi}(2x_1-k_1,2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}\tilde{p}_{k_1,k_2,k_3}^{+,-,+}\tilde{\varphi}(2x_1-k_1,k_2-2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}\tilde{p}_{k_1,k_2,k_3}^{+,-,-}\tilde{\varphi}(2x_1-k_1,k_2-2x_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}\tilde{p}_{k_1,k_2,k_3}^{-,+,+}\tilde{\varphi}(k_1-2x_1,2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}\tilde{p}_{k_1,k_2,k_3}^{-,+,-}\tilde{\varphi}(k_1-2x_1,2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}\tilde{p}_{k_1,k_2,k_3}^{-,-,+}\tilde{\varphi}(k_1-2x_1,k_2-2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}\tilde{p}_{k_1,k_2,k_3}^{-,-,-}\tilde{\varphi}(k_1-2x_1,k_2-2x_2,k_3-2x_3)
\end{aligned} \tag{10}$$

Then it can be called the sequence: $\left\{p_{k_1,k_2,k_3}^{+,+,+}\right\}_{k_1,k_2,k_3\in Z}$ and $\left\{\tilde{p}_{k_1,k_2,k_3}^{+,+,+}\right\}_{k_1,k_2,k_3\in Z}$ are three positive mask symbols; $\left\{p_{k_1,k_2,k_3}^{+,+,-}\right\}_{k_1,k_2,k_3\in Z}$ and $\left\{\tilde{p}_{k_1,k_2,k_3}^{+,+,-}\right\}_{k_1,k_2,k_3\in Z}$ are positive, positive and negative mask symbols; $\left\{p_{k_1,k_2,k_3}^{+,-,+}\right\}_{k_1,k_2,k_3\in Z}$ and $\left\{\tilde{p}_{k_1,k_2,k_3}^{+,-,+}\right\}_{k_1,k_2,k_3\in Z}$ are positive, negative and positive mask symbols; $\left\{p_{k_1,k_2,k_3}^{+,-,-}\right\}_{k_1,k_2,k_3\in Z}$ and $\left\{\tilde{p}_{k_1,k_2,k_3}^{+,-,-}\right\}_{k_1,k_2,k_3\in Z}$ are positive, negative and negative mask symbols; $\left\{p_{k_1,k_2,k_3}^{-,+,+}\right\}_{k_1,k_2,k_3\in Z}$ and $\left\{\tilde{p}_{k_1,k_2,k_3}^{-,+,+}\right\}_{k_1,k_2,k_3\in Z}$ are negative, positive, positive mask symbols; $\left\{p_{k_1,k_2,k_3}^{-,+,-}\right\}_{k_1,k_2,k_3\in Z}$ and $\left\{\tilde{p}_{k_1,k_2,k_3}^{-,+,-}\right\}_{k_1,k_2,k_3\in Z}$ are negative, positive, negative mask symbols; $\left\{p_{k_1,k_2,k_3}^{-,-,+}\right\}_{k_1,k_2,k_3\in Z}$ and $\left\{\tilde{p}_{k_1,k_2,k_3}^{-,-,+}\right\}_{k_1,k_2,k_3\in Z}$ are negative, negative, positive mask symbols; $\left\{p_{k_1,k_2,k_3}^{-,-,-}\right\}_{k_1,k_2,k_3\in Z}$ and $\left\{\tilde{p}_{k_1,k_2,k_3}^{-,-,-}\right\}_{k_1,k_2,k_3\in Z}$ are three negative mask symbols. Fourier transform of equations (9) and (10) can be obtained:

$$\begin{cases} \hat{\varphi}(\omega_1,\omega_2,\omega_3) = p^{+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + p^{+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + p^{-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + p^{-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \hat{\tilde{\varphi}}(\omega_1,\omega_2,\omega_3) = \hat{\tilde{p}}^{+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{p}}^{+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{p}}^{-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{p}}^{-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \end{cases} \tag{11}$$

There in:

$$p^{+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} p_{k_1,k_2,k_3}^{+,+,+} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};\ p^{+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} p_{k_1,k_2,k_3}^{+,+,-} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

$$p^{+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} p_{k_1,k_2,k_3}^{+,-,+} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};\ p^{+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} p_{k_1,k_2,k_3}^{+,-,-} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

$$p^{-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} p_{k_1,k_2,k_3}^{-,+,+} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};\ p^{-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} p_{k_1,k_2,k_3}^{-,+,-} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

$$p^{-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} p_{k_1,k_2,k_3}^{-,-,+} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};\ p^{-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} p_{k_1,k_2,k_3}^{-,-,-} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

The deformation of equations (9) and (10) is as follows:

$$\begin{cases}\varphi(-x_1,x_2,x_3)=\sum\limits_{k_1,k_2,k_3\in Z}p^{+,+,+}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,2x_2-k_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{+,+,-}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,2x_2-k_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}p^{+,-,+}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,k_2-2x_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{+,-,-}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,k_2-2x_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,+,+}_{k_1,k_2,k_3}\varphi(k_1+2x_1,2x_2-k_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,+,-}_{k_1,k_2,k_3}\varphi(k_1+2x_1,2x_2-k_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,-,+}_{k_1,k_2,k_3}\varphi(k_1+2x_1,k_2-2x_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,-,-}_{k_1,k_2,k_3}\varphi(k_1+2x_1,k_2-2x_2,k_3-2x_3)\\ \tilde{\varphi}(-x_1,x_2,x_3)=\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,2x_2-k_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,2x_2-k_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,k_2-2x_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,k_2-2x_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,2x_2-k_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,2x_2-k_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,k_2-2x_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,k_2-2x_2,k_3-2x_3)\end{cases}\tag{12}$$

$$\begin{cases}\varphi(x_1,-x_2,x_3)=\sum\limits_{k_1,k_2,k_3\in Z}p^{+,+,+}_{k_1,k_2,k_3}\varphi(2x_1-k_1,-2x_2-k_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{+,+,-}_{k_1,k_2,k_3}\varphi(2x_1-k_1,-2x_2-k_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}p^{+,-,+}_{k_1,k_2,k_3}\varphi(2x_1-k_1,k_2+2x_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{+,-,-}_{k_1,k_2,k_3}\varphi(2x_1-k_1,k_2+2x_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,+,+}_{k_1,k_2,k_3}\varphi(k_1-2x_1,-2x_2-k_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,+,-}_{k_1,k_2,k_3}\varphi(k_1-2x_1,-2x_2-k_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,-,+}_{k_1,k_2,k_3}\varphi(k_1-2x_1,k_2+2x_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,-,-}_{k_1,k_2,k_3}\varphi(k_1-2x_1,k_2+2x_2,k_3-2x_3)\\ \tilde{\varphi}(x_1,-x_2,x_3)=\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,-2x_2-k_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,-2x_2-k_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,k_2+2x_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,k_2+2x_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,-2x_2-k_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,-2x_2-k_2,k_3-2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,k_2+2x_2,2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,k_2+2x_2,k_3-2x_3)\end{cases}\tag{13}$$

$$\begin{cases}\varphi(x_1,x_2,-x_3)=\sum\limits_{k_1,k_2,k_3\in Z}p^{+,+,+}_{k_1,k_2,k_3}\varphi(2x_1-k_1,2x_2-k_2,-2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{+,+,-}_{k_1,k_2,k_3}\varphi(2x_1-k_1,2x_2-k_2,k_3+2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}p^{+,-,+}_{k_1,k_2,k_3}\varphi(2x_1-k_1,k_2-2x_2,-2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{+,-,-}_{k_1,k_2,k_3}\varphi(2x_1-k_1,k_2-2x_2,k_3+2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,+,+}_{k_1,k_2,k_3}\varphi(k_1-2x_1,2x_2-k_2,-2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,+,-}_{k_1,k_2,k_3}\varphi(k_1-2x_1,2x_2-k_2,k_3+2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,-,+}_{k_1,k_2,k_3}\varphi(k_1-2x_1,k_2-2x_2,-2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}p^{-,-,-}_{k_1,k_2,k_3}\varphi(k_1-2x_1,k_2-2x_2,k_3+2x_3)\\ \tilde{\varphi}(x_1,x_2,-x_3)=\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,2x_2-k_2,-2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,2x_2-k_2,k_3+2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,k_2-2x_2,-2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{+,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,k_2-2x_2,k_3+2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,2x_2-k_2,-2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,2x_2-k_2,k_3+2x_3)\\ \quad+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,k_2-2x_2,-2x_3-k_3)+\sum\limits_{k_1,k_2,k_3\in Z}\tilde{p}^{-,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,k_2-2x_2,k_3+2x_3)\end{cases}\tag{14}$$

$$\left\{\begin{aligned}
\varphi(-x_1,-x_2,x_3) &= \sum_{k_1,k_2,k_3\in Z} p^{+,+,+}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,-2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{+,+,-}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,-2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} p^{+,-,+}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,k_2+2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{+,-,-}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,k_2+2x_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} p^{-,+,+}_{k_1,k_2,k_3}\varphi(k_1+2x_1,-2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{-,+,-}_{k_1,k_2,k_3}\varphi(k_1+2x_1,-2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} p^{-,-,+}_{k_1,k_2,k_3}\varphi(k_1+2x_1,k_2+2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{-,-,-}_{k_1,k_2,k_3}\varphi(k_1+2x_1,k_2+2x_2,k_3-2x_3)\\
\tilde{\varphi}(-x_1,-x_2,x_3) &= \sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,-2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,-2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,k_2+2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,k_2+2x_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,-2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,-2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,k_2+2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,k_2+2x_2,k_3-2x_3)
\end{aligned}\right. \tag{15}$$

$$\left\{\begin{aligned}
\varphi(-x_1,x_2,-x_3) &= \sum_{k_1,k_2,k_3\in Z} p^{+,+,+}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,2x_2-k_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{+,+,-}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,2x_2-k_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} p^{+,-,+}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,k_2-2x_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{+,-,-}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,k_2-2x_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} p^{-,+,+}_{k_1,k_2,k_3}\varphi(k_1+2x_1,2x_2-k_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{-,+,-}_{k_1,k_2,k_3}\varphi(k_1+2x_1,2x_2-k_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} p^{-,-,+}_{k_1,k_2,k_3}\varphi(k_1+2x_1,k_2-2x_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{-,-,-}_{k_1,k_2,k_3}\varphi(k_1+2x_1,k_2-2x_2,k_3+2x_3)\\
\tilde{\varphi}(-x_1,x_2,-x_3) &= \sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,2x_2-k_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,2x_2-k_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,k_2-2x_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,k_2-2x_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,2x_2-k_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,2x_2-k_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,k_2-2x_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,k_2-2x_2,k_3+2x_3)
\end{aligned}\right. \tag{16}$$

$$\left\{\begin{aligned}
\varphi(x_1,-x_2,-x_3) &= \sum_{k_1,k_2,k_3\in Z} p^{+,+,+}_{k_1,k_2,k_3}\varphi(2x_1-k_1,-2x_2-k_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{+,+,-}_{k_1,k_2,k_3}\varphi(2x_1-k_1,-2x_2-k_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} p^{+,-,+}_{k_1,k_2,k_3}\varphi(2x_1-k_1,k_2+2x_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{+,-,-}_{k_1,k_2,k_3}\varphi(2x_1-k_1,k_2+2x_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} p^{-,+,+}_{k_1,k_2,k_3}\varphi(k_1-2x_1,-2x_2-k_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{-,+,-}_{k_1,k_2,k_3}\varphi(k_1-2x_1,-2x_2-k_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} p^{-,-,+}_{k_1,k_2,k_3}\varphi(k_1-2x_1,k_2+2x_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} p^{-,-,-}_{k_1,k_2,k_3}\varphi(k_1-2x_1,k_2+2x_2,k_3+2x_3)\\
\tilde{\varphi}(x_1,-x_2,-x_3) &= \sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,-2x_2-k_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,-2x_2-k_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,k_2+2x_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,k_2+2x_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,-2x_2-k_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,-2x_2-k_2,k_3+2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,k_2+2x_2,-2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,k_2+2x_2,k_3+2x_3)
\end{aligned}\right. \tag{17}$$

$$\begin{cases} \varphi(-x_1,-x_2,-x_3) = \sum_{k_1,k_2,k_3\in Z} p^{+,+,+}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,-2x_2-k_2,-2x_3-k_3) + \sum_{k_1,k_2,k_3\in Z} p^{+,+,-}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,-2x_2-k_2,k_3+2x_3) \\ \quad + \sum_{k_1,k_2,k_3\in Z} p^{+,-,+}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,k_2+2x_2,-2x_3-k_3) + \sum_{k_1,k_2,k_3\in Z} p^{+,-,-}_{k_1,k_2,k_3}\varphi(-2x_1-k_1,k_2+2x_2,k_3+2x_3) \\ \quad + \sum_{k_1,k_2,k_3\in Z} p^{-,+,+}_{k_1,k_2,k_3}\varphi(k_1+2x_1,-2x_2-k_2,-2x_3-k_3) + \sum_{k_1,k_2,k_3\in Z} p^{-,+,-}_{k_1,k_2,k_3}\varphi(k_1+2x_1,-2x_2-k_2,k_3+2x_3) \\ \quad + \sum_{k_1,k_2,k_3\in Z} p^{-,-,+}_{k_1,k_2,k_3}\varphi(k_1+2x_1,k_2+2x_2,-2x_3-k_3) + \sum_{k_1,k_2,k_3\in Z} p^{-,-,-}_{k_1,k_2,k_3}\varphi(k_1+2x_1,k_2+2x_2,k_3+2x_3) \\ \tilde{\varphi}(-x_1,-x_2,-x_3) = \sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,-2x_2-k_2,-2x_3-k_3) + \sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,-2x_2-k_2,k_3+2x_3) \\ \quad + \sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,k_2+2x_2,-2x_3-k_3) + \sum_{k_1,k_2,k_3\in Z} \tilde{p}^{+,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(-2x_1-k_1,k_2+2x_2,k_3+2x_3) \\ \quad + \sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,-2x_2-k_2,-2x_3-k_3) + \sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,-2x_2-k_2,k_3+2x_3) \\ \quad + \sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,k_2+2x_2,-2x_3-k_3) + \sum_{k_1,k_2,k_3\in Z} \tilde{p}^{-,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1+2x_1,k_2+2x_2,k_3+2x_3) \end{cases} \tag{18}$$

Fourier transform of (12) ~ (18) can be obtained as follows:

$$\begin{cases} \hat{\varphi}(-\omega_1,\omega_2,\omega_3) = p^{+,+,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{+,+,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + p^{+,-,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{+,-,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + p^{-,+,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{-,+,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + p^{-,-,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{-,-,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \hat{\tilde{\varphi}}(-\omega_1,\omega_2,\omega_3) = \hat{\tilde{p}}^{+,+,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{+,+,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{p}}^{+,-,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{+,-,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{p}}^{-,+,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{-,+,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{p}}^{-,-,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{-,-,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \end{cases} \tag{19}$$

$$\begin{cases} \hat{\varphi}(\omega_1,-\omega_2,\omega_3) = p^{+,+,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{+,+,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + p^{+,-,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{+,-,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + p^{-,+,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{-,+,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + p^{-,-,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + p^{-,-,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \hat{\tilde{\varphi}}(\omega_1,-\omega_2,\omega_3) = \hat{\tilde{p}}^{+,+,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{+,+,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{p}}^{+,-,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{+,-,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{p}}^{-,+,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{-,+,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{p}}^{-,-,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{p}}^{-,-,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \end{cases} \tag{20}$$

$$\begin{cases}
\hat{\varphi}(\omega_1,\omega_2,-\omega_3)=p^{+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})+p^{+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
\quad +p^{+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})+p^{+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
\quad +p^{-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})+p^{-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
\quad +p^{-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})+p^{-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
\hat{\tilde{\varphi}}(\omega_1,\omega_2,-\omega_3)=\hat{\tilde{p}}^{+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})+\hat{\tilde{p}}^{+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
\quad +\hat{\tilde{p}}^{+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})+\hat{\tilde{p}}^{+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
\quad +\hat{\tilde{p}}^{-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})+\hat{\tilde{p}}^{-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
\quad +\hat{\tilde{p}}^{-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})+\hat{\tilde{p}}^{-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})
\end{cases} \tag{21}$$

$$\begin{cases}
\hat{\varphi}(-\omega_1,-\omega_2,\omega_3)=p^{+,+,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})+p^{+,+,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\
\quad +p^{+,-,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})+p^{+,-,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\
\quad +p^{-,+,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})+p^{-,+,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\
\quad +p^{-,-,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})+p^{-,-,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\
\hat{\tilde{\varphi}}(-\omega_1,-\omega_2,\omega_3)=\hat{\tilde{p}}^{+,+,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})+\hat{\tilde{p}}^{+,+,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\
\quad +\hat{\tilde{p}}^{+,-,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})+\hat{\tilde{p}}^{+,-,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\
\quad +\hat{\tilde{p}}^{-,+,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})+\hat{\tilde{p}}^{-,+,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\
\quad +\hat{\tilde{p}}^{-,-,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})+\hat{\tilde{p}}^{-,-,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})
\end{cases} \tag{22}$$

$$\begin{cases}
\hat{\varphi}(-\omega_1,\omega_2,-\omega_3)= \\
p^{+,+,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})+p^{+,+,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
+p^{+,-,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})+p^{+,-,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
+p^{-,+,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})+p^{-,+,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
+p^{-,-,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})+p^{-,-,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
\hat{\tilde{\varphi}}(-\omega_1,\omega_2,-\omega_3)= \\
\hat{\tilde{p}}^{+,+,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})+\hat{\tilde{p}}^{+,+,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
+\hat{\tilde{p}}^{+,-,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})+\hat{\tilde{p}}^{+,-,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
+\hat{\tilde{p}}^{-,+,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})+\hat{\tilde{p}}^{-,+,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
+\hat{\tilde{p}}^{-,-,+}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})+\hat{\tilde{p}}^{-,-,-}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2})
\end{cases} \tag{23}$$

$$\left\{\begin{aligned}
\hat{\varphi}(\omega_1,-\omega_2,-\omega_3) = {} & p^{+,+,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) + p^{+,+,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + p^{+,-,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) + p^{+,-,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + p^{-,+,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) + p^{-,+,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + p^{-,-,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) + p^{-,-,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
\hat{\tilde{\varphi}}(\omega_1,-\omega_2,-\omega_3) = {} & \hat{\tilde{p}}^{+,+,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) + \hat{\tilde{p}}^{+,+,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + \hat{\tilde{p}}^{+,-,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) + \hat{\tilde{p}}^{+,-,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + \hat{\tilde{p}}^{-,+,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) + \hat{\tilde{p}}^{-,+,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + \hat{\tilde{p}}^{-,-,+}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) + \hat{\tilde{p}}^{-,-,-}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})
\end{aligned}\right. \tag{24}$$

$$\left\{\begin{aligned}
\hat{\varphi}(-\omega_1,-\omega_2,-\omega_3) = {} & p^{+,+,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) + p^{+,+,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + p^{+,-,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) + p^{+,-,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + p^{-,+,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) + p^{-,+,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + p^{-,-,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) + p^{-,-,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
\hat{\tilde{\varphi}}(-\omega_1,-\omega_2,-\omega_3) = {} & \hat{\tilde{p}}^{+,+,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) + \hat{\tilde{p}}^{+,+,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + \hat{\tilde{p}}^{+,-,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) + \hat{\tilde{p}}^{+,-,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + \hat{\tilde{p}}^{-,+,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) + \hat{\tilde{p}}^{-,+,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
& + \hat{\tilde{p}}^{-,-,+}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) + \hat{\tilde{p}}^{-,-,-}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})
\end{aligned}\right. \tag{25}$$

If the:

$$\begin{aligned}
\phi(x_1,x_2,x_3) = {} & [\varphi(x_1,x_2,x_3),\varphi(-x_1,x_2,x_3),\varphi(x_1,-x_2,x_3),\varphi(x_1,x_2,-x_3), \\
& \varphi(-x_1,-x_2,x_3),\varphi(-x_1,x_2,-x_3),\varphi(x_1,-x_2,-x_3),\varphi(-x_1,-x_2,-x_3)]^{\mathrm{T}}
\end{aligned} \tag{26}$$

$$\begin{aligned}
\tilde{\phi}(x_1,x_2,x_3) = {} & [\tilde{\varphi}(x_1,x_2,x_3),\tilde{\varphi}(-x_1,x_2,x_3),\tilde{\varphi}(x_1,-x_2,x_3),\tilde{\varphi}(x_1,x_2,-x_3), \\
& \tilde{\varphi}(-x_1,-x_2,x_3),\tilde{\varphi}(-x_1,x_2,-x_3),\tilde{\varphi}(x_1,-x_2,-x_3),\tilde{\varphi}(-x_1,-x_2,-x_3)]^{\mathrm{T}}
\end{aligned} \tag{27}$$

Where 'T' represents the transpose of the matrix, and there can be partitioned matrices:

$$A = \begin{pmatrix}
p^{+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
p^{+,+,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{+,+,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{-,-,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{-,+,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\
p^{+,-,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{-,-,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{+,+,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{-,-,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) \\
p^{+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2})
\end{pmatrix}$$

$$B=\begin{pmatrix} p^{-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\ p^{+,-,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{-,+,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{+,-,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & p^{-,-,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\ p^{+,+,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{+,-,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{-,+,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{-,+,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) \\ p^{-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) \end{pmatrix}$$

$$C=\begin{pmatrix} p^{-,+,+}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{+,-,+}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{+,+,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{-,-,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) \\ p^{-,+,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{-,+,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,-,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,+,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) \\ p^{-,-,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,-,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{-,+,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,-,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) \\ p^{-,-,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{-,-,+}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{-,+,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{-,+,+}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) \end{pmatrix}$$

$$D=\begin{pmatrix} p^{+,+,+}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{-,-,+}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{-,+,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & p^{+,-,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) \\ p^{-,-,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,+,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{-,-,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,-,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) \\ p^{-,+,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{-,-,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,+,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,+,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) \\ p^{+,-,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,-,+}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,+,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & p^{+,+,+}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) \end{pmatrix}$$

Thus, the symbols of the enhanced mask in Equations (26) and (27) can be obtained.

3. Biorthogonal of the Two-scale Three-dimensional Eight-direction Scale Functions and Wavelet Functions

Definition 3.1 If the three-dimensional eight-direction scale function $\varphi(x_1,x_2,x_3)$ [16] is orthogonal, then the following conditions are true :

$$\begin{aligned}
&\left\langle \varphi(x_1,x_2,x_3),\varphi(x_1-k_1,x_2-k_2,x_3-k_3)\right\rangle=\delta_{0,k_1}\delta_{0,k_2}\delta_{0,k_3};\\
&\left\langle \varphi(x_1,x_2,x_3),\varphi(k_1-x_1,x_2-k_2,x_3-k_3)\right\rangle=0;\\
&\left\langle \varphi(x_1,x_2,x_3),\varphi(x_1-k_1,k_2-x_2,x_3-k_3)\right\rangle=0;\\
&\left\langle \varphi(x_1,x_2,x_3),\varphi(x_1-k_1,x_2-k_2,k_3-x_3)\right\rangle=0;\\
&\left\langle \varphi(x_1,x_2,x_3),\varphi(k_1-x_1,k_2-x_2,x_3-k_3)\right\rangle=0;\\
&\left\langle \varphi(x_1,x_2,x_3),\varphi(k_1-x_1,x_2-k_2,k_3-x_3)\right\rangle=0;\\
&\left\langle \varphi(x_1,x_2,x_3),\varphi(x_1-k_1,k_2-x_2,k_3-x_3)\right\rangle=0;\\
&\left\langle \varphi(x_1,x_2,x_3),\varphi(k_1-x_1,k_2-x_2,k_3-x_3)\right\rangle=0;
\end{aligned}\tag{28}$$

Definition 3.2 If the three-dimensional eight-direction scale functions $\varphi(x_1,x_2,x_3)$ and $\tilde{\varphi}(x_1,x_2,x_3)$ [16] are biorthogonal, then the following equations are satisfied :

$$\begin{aligned}
&\langle \varphi(x_1,x_2,x_3), \tilde{\varphi}(x_1-k_1,x_2-k_2,x_3-k_3)\rangle = \delta_{0,k_1}\delta_{0,k_2}\delta_{0,k_3};\\
&\langle \varphi(x_1,x_2,x_3), \tilde{\varphi}(k_1-x_1,x_2-k_2,x_3-k_3)\rangle = 0;\\
&\langle \varphi(x_1,x_2,x_3), \tilde{\varphi}(x_1-k_1,k_2-x_2,x_3-k_3)\rangle = 0;\\
&\langle \varphi(x_1,x_2,x_3), \tilde{\varphi}(x_1-k_1,x_2-k_2,k_3-x_3)\rangle = 0;\\
&\langle \varphi(x_1,x_2,x_3), \tilde{\varphi}(k_1-x_1,k_2-x_2,x_3-k_3)\rangle = 0;\\
&\langle \varphi(x_1,x_2,x_3), \tilde{\varphi}(k_1-x_1,x_2-k_2,k_3-x_3)\rangle = 0;\\
&\langle \varphi(x_1,x_2,x_3), \tilde{\varphi}(x_1-k_1,k_2-x_2,k_3-x_3)\rangle = 0;\\
&\langle \varphi(x_1,x_2,x_3), \tilde{\varphi}(k_1-x_1,k_2-x_2,k_3-x_3)\rangle = 0;
\end{aligned} \tag{29}$$

Definition 3.3 If $\{V_j\}_{j\in Z}$ is a three-dimensional eight-direction multi-resolution analysis of $\varphi(x_1,x_2,x_3)$ scale function and $\{W_j\}_{j\in Z}$ is the orthogonal complement space of V_j in V_{j+1} as well as $L^2(R^3) = \oplus_{j\in Z} W_j$, where $W_j = W_j^{(1)} \oplus W_j^{(2)} \oplus W_j^{(3)} \oplus W_j^{(4)} \oplus W_j^{(5)} \oplus W_j^{(6)} \oplus W_j^{(7)}$, $W_j^{(\gamma)}(\gamma = 1,2,3,4,5,6,7)$ is orthogonal to each other. Thus, there exists a ternary compact supported wavelet function $\psi^\gamma(x_1,x_2,x_3): \gamma = 1,2,3,4,5,6,7$, making the function set:

$$\begin{aligned}
&\{\psi^\gamma(x_1-k_1,x_2-k_2,x_3-k_3), \psi^\gamma(k_1-x_1,x_2-k_2,x_3-k_3),\\
&\psi^\gamma(x_1-k_1,k_2-x_2,x_3-k_3), \psi^\gamma(x_1-k_1,x_2-k_2,k_3-x_3),\\
&\psi^\gamma(k_1-x_1,k_2-x_2,x_3-k_3), \psi^\gamma(k_1-x_1,x_2-k_2,k_3-x_3),\\
&\psi^\gamma(x_1-k_1,k_2-x_2,k_3-x_3), \psi^\gamma(k_1-x_1,k_2-x_2,k_3-x_3)\\
&: \gamma = 1,2,3,4,5,6,7; k_1,k_2,k_3 \in Z\}
\end{aligned}$$

It forms an orthonormal basis for W_0. The tightly supported wavelet function $\tilde{\psi}^\gamma(x_1,x_2,x_3): \gamma = 1,2,3,4,5,6,7.$ corresponding to $\psi^\gamma(x_1,x_2,x_3): \gamma = 1,2,3,4,5,6,7.$ satisfies:

$$\begin{aligned}
&\left\langle \varphi(x_1,x_2,x_3), \psi^\gamma(x_1-k_1,x_2-k_2,x_3-k_3)\right\rangle = \left\langle \varphi(x_1,x_2,x_3), \psi^\gamma(k_1-x_1,x_2-k_2,x_3-k_3)\right\rangle = 0;\\
&\left\langle \tilde{\varphi}(x_1,x_2,x_3), \psi^\gamma(x_1-k_1,x_2-k_2,x_3-k_3)\right\rangle = \left\langle \tilde{\varphi}(x_1,x_2,x_3), \psi^\gamma(k_1-x_1,x_2-k_2,x_3-k_3)\right\rangle = 0;\\
&\left\langle \varphi(x_1,x_2,x_3), \psi^\gamma(x_1-k_1,k_2-x_2,x_3-k_3)\right\rangle = \left\langle \varphi(x_1,x_2,x_3), \psi^\gamma(x_1-k_1,x_2-k_2,k_3-x_3)\right\rangle = 0;\\
&\left\langle \tilde{\varphi}(x_1,x_2,x_3), \psi^\gamma(x_1-k_1,k_2-x_2,x_3-k_3)\right\rangle = \left\langle \tilde{\varphi}(x_1,x_2,x_3), \psi^\gamma(x_1-k_1,x_2-k_2,k_3-x_3)\right\rangle = 0;\\
&\left\langle \varphi(x_1,x_2,x_3), \psi^\gamma(k_1-x_1,k_2-x_2,x_3-k_3)\right\rangle = \left\langle \varphi(x_1,x_2,x_3), \psi^\gamma(k_1-x_1,x_2-k_2,k_3-x_3)\right\rangle = 0;\\
&\left\langle \tilde{\varphi}(x_1,x_2,x_3), \psi^\gamma(k_1-x_1,k_2-x_2,x_3-k_3)\right\rangle = \left\langle \tilde{\varphi}(x_1,x_2,x_3), \psi^\gamma(k_1-x_1,x_2-k_2,k_3-x_3)\right\rangle = 0;\\
&\left\langle \varphi(x_1,x_2,x_3), \psi^\gamma(x_1-k_1,k_2-x_2,k_3-x_3)\right\rangle = \left\langle \varphi(x_1,x_2,x_3), \psi^\gamma(k_1-x_1,k_2-x_2,k_3-x_3)\right\rangle = 0;\\
&\left\langle \tilde{\varphi}(x_1,x_2,x_3), \psi^\gamma(x_1-k_1,k_2-x_2,k_3-x_3)\right\rangle = \left\langle \tilde{\varphi}(x_1,x_2,x_3), \psi^\gamma(k_1-x_1,k_2-x_2,k_3-x_3)\right\rangle = 0;
\end{aligned}$$

$$\begin{aligned}
&\left\langle \psi^k(x_1,x_2,x_3),\psi^t(x_1-k_1,x_2-k_2,x_3-k_3)\right\rangle=\delta_{k,t}\delta_{0,k_1}\delta_{0,k_2}\delta_{0,k_3};\left\langle \psi^k(x_1,x_2,x_3),\psi^t(k_1-x_1,x_2-k_2,x_3-k_3)\right\rangle=0;\\
&\left\langle \tilde{\psi}^k(x_1,x_2,x_3),\tilde{\psi}^t(x_1-k_1,x_2-k_2,x_3-k_3)\right\rangle=\delta_{k,t}\delta_{0,k_1}\delta_{0,k_2}\delta_{0,k_3};\left\langle \tilde{\psi}^k(x_1,x_2,x_3),\tilde{\psi}^t(k_1-x_1,x_2-k_2,x_3-k_3)\right\rangle=0;\\
&\left\langle \psi^k(x_1,x_2,x_3),\psi^t(x_1-k_1,k_2-x_2,x_3-k_3)\right\rangle=0;\left\langle \psi^k(x_1,x_2,x_3),\psi^t(x_1-k_1,x_2-k_2,k_3-x_3)\right\rangle=0;\\
&\left\langle \tilde{\psi}^k(x_1,x_2,x_3),\tilde{\psi}^t(x_1-k_1,k_2-x_2,x_3-k_3)\right\rangle=0;\left\langle \tilde{\psi}^k(x_1,x_2,x_3),\tilde{\psi}^t(x_1-k_1,x_2-k_2,k_3-x_3)\right\rangle=0;\\
&\left\langle \psi^k(x_1,x_2,x_3),\psi^t(k_1-x_1,k_2-x_2,x_3-k_3)\right\rangle=0;\left\langle \psi^k(x_1,x_2,x_3),\psi^t(k_1-x_1,x_2-k_2,k_3-x_3)\right\rangle=0;\\
&\left\langle \tilde{\psi}^k(x_1,x_2,x_3),\tilde{\psi}^t(k_1-x_1,k_2-x_2,x_3-k_3)\right\rangle=0;\left\langle \tilde{\psi}^k(x_1,x_2,x_3),\tilde{\psi}^t(k_1-x_1,x_2-k_2,k_3-x_3)\right\rangle=0;\\
&\left\langle \psi^k(x_1,x_2,x_3),\psi^t(x_1-k_1,k_2-x_2,k_3-x_3)\right\rangle=0;\left\langle \psi^k(x_1,x_2,x_3),\psi^t(k_1-x_1,k_2-x_2,k_3-x_3)\right\rangle=0;\\
&\left\langle \tilde{\psi}^k(x_1,x_2,x_3),\tilde{\psi}^t(x_1-k_1,k_2-x_2,k_3-x_3)\right\rangle=0;\left\langle \tilde{\psi}^k(x_1,x_2,x_3),\tilde{\psi}^t(k_1-x_1,k_2-x_2,k_3-x_3)\right\rangle=0;
\end{aligned} \tag{30}$$

Among them: $\gamma,k,t=1,2,3,4,5,6,7.$

Hence there is a sequence:

$$\begin{aligned}
&\Big(\left\{q^{\gamma+,+,+}_{k_1,k_2,k_3}\right\}_{k_1,k_2,k_3\in Z},\left\{q^{\gamma+,+,-}_{k_1,k_2,k_3}\right\}_{k_1,k_2,k_3\in Z},\\
&\left\{q^{\gamma+,-,+}_{k_1,k_2,k_3}\right\}_{k_1,k_2,k_3\in Z},\left\{q^{\gamma+,-,-}_{k_1,k_2,k_3}\right\}_{k_1,k_2,k_3\in Z},\\
&\left\{q^{\gamma-,+,+}_{k_1,k_2,k_3}\right\}_{k_1,k_2,k_3\in Z},\left\{q^{\gamma-,+,-}_{k_1,k_2,k_3}\right\}_{k_1,k_2,k_3\in Z},\\
&\left\{q^{\gamma-,-,+}_{k_1,k_2,k_3}\right\}_{k_1,k_2,k_3\in Z},\left\{q^{\gamma-,-,-}_{k_1,k_2,k_3}\right\}_{k_1,k_2,k_3\in Z}\Big)\in l^2(Z^3)
\end{aligned}$$

To satisfy:

$$\begin{aligned}
\psi^\gamma(x_1,x_2,x_3)=&\sum_{k_1,k_2,k_3\in Z}q^{\gamma+,+,+}_{k_1,k_2,k_3}\varphi(2x_1-k_1,2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}q^{\gamma+,+,-}_{k_1,k_2,k_3}\varphi(2x_1-k_1,2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}q^{\gamma+,-,+}_{k_1,k_2,k_3}\varphi(2x_1-k_1,k_2-2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}q^{\gamma+,-,-}_{k_1,k_2,k_3}\varphi(2x_1-k_1,k_2-2x_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}q^{\gamma-,+,+}_{k_1,k_2,k_3}\varphi(k_1-2x_1,2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}q^{\gamma-,+,-}_{k_1,k_2,k_3}\varphi(k_1-2x_1,2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}q^{\gamma-,-,+}_{k_1,k_2,k_3}\varphi(k_1-2x_1,k_2-2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}q^{\gamma-,-,-}_{k_1,k_2,k_3}\varphi(k_1-2x_1,k_2-2x_2,k_3-2x_3)
\end{aligned} \tag{31}$$

The same is true for $\tilde{\psi}^\gamma(x_1,x_2,x_3):\gamma=1,2,3,4,5,6,7.$:

$$\begin{aligned}
\tilde{\psi}^\gamma(x_1,x_2,x_3)=&\sum_{k_1,k_2,k_3\in Z}\tilde{q}^{\gamma+,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}\tilde{q}^{\gamma+,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}\tilde{q}^{\gamma+,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,k_2-2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}\tilde{q}^{\gamma+,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(2x_1-k_1,k_2-2x_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}\tilde{q}^{\gamma-,+,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,2x_2-k_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}\tilde{q}^{\gamma-,+,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,2x_2-k_2,k_3-2x_3)\\
&+\sum_{k_1,k_2,k_3\in Z}\tilde{q}^{\gamma-,-,+}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,k_2-2x_2,2x_3-k_3)+\sum_{k_1,k_2,k_3\in Z}\tilde{q}^{\gamma-,-,-}_{k_1,k_2,k_3}\tilde{\varphi}(k_1-2x_1,k_2-2x_2,k_3-2x_3)
\end{aligned} \tag{32}$$

The Fourier transform of (31) and (32) can be obtained:

$$\begin{cases} \hat{\psi}^{\gamma}(\omega_1,\omega_2,\omega_3) = Q^{\gamma+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + Q^{\gamma+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + Q^{\gamma+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + Q^{\gamma+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + Q^{\gamma-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + Q^{\gamma-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + Q^{\gamma-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + Q^{\gamma-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\varphi}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \hat{\tilde{\psi}}^{\gamma}(\omega_1,\omega_2,\omega_3) = \hat{\tilde{Q}}^{\gamma+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{Q}}^{\gamma+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{Q}}^{\gamma+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{Q}}^{\gamma+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{Q}}^{\gamma-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{Q}}^{\gamma-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}) \\ \quad + \hat{\tilde{Q}}^{\gamma-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}) + \hat{\tilde{Q}}^{\gamma-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2})\hat{\tilde{\varphi}}(-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}) \end{cases} \tag{33}$$

Among them:

$$Q^{\gamma+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} q^{\gamma+,+,+}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})}; Q^{\gamma+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} q^{\gamma+,+,-}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

$$Q^{\gamma+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} q^{\gamma+,-,+}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})}; Q^{\gamma+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} q^{\gamma+,-,-}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

$$Q^{\gamma-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} q^{\gamma-,+,+}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})}; Q^{\gamma-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} q^{\gamma-,+,-}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

$$Q^{\gamma-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} q^{\gamma-,-,+}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})}; Q^{\gamma-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} q^{\gamma-,-,-}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

$\psi^{\gamma}(x_1,x_2,x_3)$ is called the three positive mask symbols; positive, positive and negative mask symbols; positive, negative and positive mask symbols; positive, negative and negative mask symbols; negative, positive and positive mask symbols; negative, positive and negative mask symbols; negative, negative and positive mask symbols; three negative mask symbols, respectively. The corresponding $\tilde{\psi}^{\gamma}(x_1,x_2,x_3):\gamma=1,2,3,4,5,6,7$ are three positive mask symbols; positive, positive and negative mask symbols; positive, negative and positive mask symbols; positive, negative and negative mask symbols; negative, positive and positive mask symbols; negative, positive and negative mask symbols; negative, negative and positive mask symbols; three negative mask symbols, respectively. As follows:

$$\tilde{Q}^{\gamma+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} \tilde{q}^{\gamma+,+,+}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})}; \tilde{Q}^{\gamma+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} \tilde{q}^{\gamma+,+,-}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

$$\tilde{Q}^{\gamma+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} \tilde{q}^{\gamma+,-,+}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})}; \tilde{Q}^{\gamma+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} \tilde{q}^{\gamma+,-,-}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

$$\tilde{Q}^{\gamma-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} \tilde{q}^{\gamma-,+,+}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})}; \tilde{Q}^{\gamma-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} \tilde{q}^{\gamma-,+,-}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

$$\tilde{Q}^{\gamma-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} \tilde{q}^{\gamma-,-,+}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})}; \tilde{Q}^{\gamma-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) = \frac{1}{8}\sum_{k_1,k_2,k_3\in Z} \tilde{q}^{\gamma-,-,-}_{k_1,k_2,k_3} e^{-i(\frac{\omega_1 k_1}{2}+\frac{\omega_2 k_2}{2}+\frac{\omega_3 k_3}{2})};$$

If the:

$$\begin{cases} \Psi^{\gamma}(x_1,x_2,x_3)=(\psi^{\gamma}(x_1,x_2,x_3),\psi^{\gamma}(-x_1,x_2,x_3),\psi^{\gamma}(x_1,-x_2,x_3),\psi^{\gamma}(x_1,x_2,-x_3), \\ \qquad \psi^{\gamma}(-x_1,-x_2,x_3),\psi^{\gamma}(-x_1,x_2,-x_3),\psi^{\gamma}(x_1,-x_2,-x_3),\psi^{\gamma}(-x_1,-x_2,-x_3))^{\mathrm{T}}; \\ \tilde{\Psi}^{\gamma}(x_1,x_2,x_3)=(\tilde{\psi}^{\gamma}(x_1,x_2,x_3),\tilde{\psi}^{\gamma}(-x_1,x_2,x_3),\tilde{\psi}^{\gamma}(x_1,-x_2,x_3),\tilde{\psi}^{\gamma}(x_1,x_2,-x_3), \\ \qquad \tilde{\psi}^{\gamma}(-x_1,-x_2,x_3),\tilde{\psi}^{\gamma}(-x_1,x_2,-x_3),\tilde{\psi}^{\gamma}(x_1,-x_2,-x_3),\tilde{\psi}^{\gamma}(-x_1,-x_2,-x_3))^{\mathrm{T}}; \end{cases}$$

Where 'T' is the transpose of the matrix, and there are partitioned matrix E,F,G and H whose specific form are as follows:

$$E=\begin{pmatrix} Q^{\gamma+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\ Q^{\gamma+,+,-}(\frac{-\omega_1}{2}\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma+,+,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,-,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\ Q^{\gamma+,-,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,-,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}_{3}) & Q^{\gamma+,+,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,-,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) \\ Q^{\gamma+,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma+,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) \end{pmatrix}$$

$$F=\begin{pmatrix} Q^{\gamma-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\ Q^{\gamma+,-,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,+,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma+,-,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,-,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}) \\ Q^{\gamma+,+,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma+,-,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,+,+}\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}_{3}) & Q^{\gamma-,+,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) \\ Q^{\gamma-,-,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma+,-,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma-,+,+}(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) \end{pmatrix}$$

$$G=\begin{pmatrix} Q^{\gamma-,+,+}(-\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma+,-,+}-\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma+,+,-}(-\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,-,-}(-\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) \\ Q^{\gamma-,+,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma-,+,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma+,-,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) \\ Q^{\gamma-,-,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma+,-,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma-,+,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}_{3}) & Q^{\gamma+,-,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) \\ Q^{\gamma-,-,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma-,-,+}\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}_{3}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma-,+,+}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) \end{pmatrix}$$

$$H=\begin{pmatrix} Q^{\gamma+,+,+}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma-,-,+}\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}_{3}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) & Q^{\gamma+,-,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{\omega_3}{2}) \\ Q^{\gamma-,-,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma+,+,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma-,-,-}\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}_{3}) & Q^{\gamma+,-,+}(\frac{-\omega_1}{2},\frac{\omega_2}{2},\frac{-\omega_3}{2}) \\ Q^{\gamma-,+,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma-,-,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma+,+,+}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) \\ Q^{\gamma+,-,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma+,-,+}\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}_{3}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) & Q^{\gamma+,+,+}(\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}) \end{pmatrix}$$

Then, the enhanced masks corresponding to $\Psi^{\gamma}(x_1,x_2,x_3)$ and $\tilde{\Psi}^{\gamma}(x_1,x_2,x_3)$ can be written as follows:

$$I^{\gamma}(\omega_1,\omega_2,\omega_3)=\begin{bmatrix} E & F \\ G & H \end{bmatrix};\ \tilde{I}^{\gamma}(\omega_1,\omega_2,\omega_3)=\begin{bmatrix} \tilde{E} & \tilde{F} \\ \tilde{G} & \tilde{H} \end{bmatrix}$$

4. Conclusion

Based on bidirectional wavelet theory and tensor product principle, the two-scale three-dimensional eight-direction multiresolution analysis is given. Furthermore, the construction of two-scale three-dimensional eight-direction scale function and wavelet function is studied, and the two-scale three-dimensional eight-direction wavelets under orthogonal and biorthogonal conditions are obtained. Hereby, the one-dimension orthogonal biorthogonal wavelet is extended to two-scale three-dimension eight-dimension biorthogonal wavelet; this research may promote the theoretical study of high dimensional wavelet.

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