American Journal of Applied Mathematics
2021; 9(2): 44-51
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20210902.12
ISSN: 2330-0043 (Print); ISSN: 2330-006X (Online)

The Fermionic-bosonic Representations of *A*(*M*-1,*N*-1)-graded Lie Superalgebras

Lingyu Yu

School of Mathematics and Statistics, Shandong Normal University, Jinan, P. R. China

Email address:
17864193324@163.com

To cite this article:
Lingyu Yu. The Fermionic-bosonic Representations of $A(M\text{-}1,N\text{-}1)$-graded Lie Superalgebras. *American Journal of Applied Mathematics*. Vol. 9, No. 2, 2021, pp. 44-51. doi: 10.11648/j.ajam.20210902.12

Received: March 17, 2021; **Accepted**: March 31, 2021; **Published**: April 10, 2021

Abstract: Lie groups, Lie algebras and their representation theories are important parts of mathematical physics. They play a crucial character in symmetries. As a generalization of Lie algebra, Lie superalgebras are from the comprehending and description for supersymmetry of mathematical physics. Unlike the semisimple Lie algebras, understanding the representation theory of Lie superalgebras is a difficult problem. Lie superalgebras graded by root supersystems are Lie superalgebras of great significance. In recent years, the representations of types $B(m,n)$, $C(n)$, $D(m,n)$, $P(n)$ and $Q(n)$-graded Lie superalgebras coordinatized by quantum tori have been studied. In this paper, we construct fermionic-bosonic representations for a class of $A(M\text{-}1,N\text{-}1)$-graded Lie superalgebras coordinatized by quantum tori with nontrivial central extensions. At first, we introduce the background of the research on the graded Lie superalgebras and present some basics on it. Then, a set of bases for $A(M\text{-}1,N\text{-}1)$-graded Lie superalgebras and the multiplication operations among them are given specifically to present the construction of the vector space. By using the tensor product of fermionic and bosonic module, the operators and their operation relations are derived. Finally, we obtain a brief and pretty representation theorem of $A(M\text{-}1,N\text{-}1)$-graded Lie superalgebras with nontrivial central extensions.

Keywords: Graded Lie Superalgebras, Quantum Tori, Fermionic-Bosonic Representations, Nontrivial Central Extensions

1. Introduction

At the end of the 19th century, Lie proposed the concept of continuous transformation group when he studied the invariance of differential equations. In memory of his contribution, futurity call it Lie group. Later, mathematicians considered its linearization at the infinitesimal level, and thus obtained a structure of infinitesimal group, which is called Lie algebra. In the 1970s, the development of bosons and fermions theory in physics prompted people to propose supersymmetry. Therefore, people introduced Lie superalgebra structure to characterize it.

Lie superalgebras are Lie algebras with Z_2 -graded structure [1], which are natural generalizations of Lie algebras. Lie superalgebra theory is more complex than Lie algebra theory. On the one hand, Lie superalgebra theory is closely related to quantum field theory, statistical mechanics and string theory, so scholars in various fields need to have a profound physical background to study Lie superalgebra; on the other hand, close contact between Lie superalgebra and many branches of mathematics, such as quantum supergroup, differential geometry, topology and algebraic groups, etc.

In 1992, Berman and Moody [2] gave the concept of Lie algebras graded by finite root systems so as to comprehend the notion of generalized intersection matrix algebras proposed by Slodowy. Since then, under the research of many scholars, complete classification of all graded Lie algebras has been realized [3-5]. Graded Lie superalgebras are natural extension of graded Lie algebras. Unlike Lie algebras, even for finite dimensional Lie superalgebras has no strict root theory. Since entering the 21st century, Benkart and Elduque have given some concepts of classical finite dimensional graded Lie superalgebras based on the root supersystems of classical Lie superalgebras given by Kac [6], and have classified types $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1;\alpha)$, $F(4)$ and $G(3)$-graded Lie superalgebras [7-9]. Gao [10] gave the fermionic and bosonic representations of the high dimensional affine Lie algebra $\widehat{gl_N(\mathbb{C}_q)}$ with quantum torus $\mathbb{C}_q$ of two variables as coordinate algebra. Later, Chen-Gao [12, 13] structured representations for BC_N -graded Lie algebras and $B(0,N)$-graded Lie superalgebras coordinatized by quantum tori. Cheng established the bosonic and fermionic representations for a kind of generalized $B(m,n)$, $C(n)$, $D(m,n)$, $P(N)$, $Q(N)$-graded Lie superalgebras with the quantum tori as the coordinate algebra [14-16]. At present, the research of Lie superalgebra is still limited, and many works are far from complete.

In the letter, by constructing fermionic-bosonic representations, one obtains $A(M-1,N-1)$ -graded Lie superalgebras with nontrivial central extensions.

The remainder is arranged as follows. Certain basics on Lie superalgebras, $A(M-1,N-1)$-graded Lie superalgebras, the quantum tori and the matrix Lie superalgebras coordinatized by quantum tori are looked back in section 2. Then, in section 3, we present the construction of $A(M-1,N-1)$-graded Lie superalgebras with the quantum tori as the coordinate algebra. And the representations for $A(M-1,N-1)$-graded Lie superalgebras coordinatized by quantum tori are established by using the tensor product of fermionic and bosonic module. Finally, section 4 presents the conclusion.

Notations: In this paper, we assume that $\mathbb{F}$ is a field of characteristic zero. One denotes $\mathbb{Z}$ as the set of integers and $\mathbb{N}^*$ as the set of positive integers.

2. Preliminaries

In this section, we recall some basics. First, we review the basic concepts of Lie superalgebras. Then we introduce Lie superalgebras $A(M-1,N-1)$ and the definition of its graded Lie superalgebras. Finally, we introduce the quantum tori and the matrix Lie superalgebras with the quantum tori as the coordinate algebra.

2.1. Notions of Lie Superalgebras

Definition 1. Vector space V is a Z_2-graded vector space if V is the direct sum of two vector spaces, i.e.

$$V=V_{\bar{0}}\oplus V_{\bar{1}}.$$

Remark 1. $v\in V_{\bar{i}}\,(i=0,1)$ is called homogeneous element and we denote $|v|=i$ as the degree of v.

Definition 2. Suppose that $G=G_{\bar{0}}\oplus G_{\bar{1}}$ is a Z_2-graded vector space over field $\mathbb{F}$ with a bilinear operation $[\cdot,\cdot]:G\times G\to G$. G is a Lie superalgebra over $\mathbb{F}$ if the three conditions are met:

$$(i)[G_\alpha,G_\beta]\subseteq G_{\alpha+\beta}\text{ for all }\alpha,\beta\text{ in }Z_2=\{\bar{0},\bar{1}\};$$
$$(ii)[x,y]=-(-1)^{|x||y|}[y,x];$$
$$(iii)[x,[y,z]]=[[x,y],z]+(-1)^{|x||y|}[y,[z,x]],$$

for all $x,y,z\in G_\alpha,\alpha\in Z_2$.

Remark 2. Let $gl(m,n)(\mathbb{F})$ (or simply as $gl(m,n)$) be a set of all block matrices over field $\mathbb{F}$ with the following form:

$$X=\begin{pmatrix} A & B\\ C & D\end{pmatrix},$$

where A and D are matrices over $\mathbb{F}$ with order $m\times m$ and $n\times n$, respectively. Then, $gl(m,n)=\begin{pmatrix} A & 0\\ 0 & D\end{pmatrix}\oplus\begin{pmatrix} 0 & B\\ C & 0\end{pmatrix}$ under the bilinear operation $[x,y]:=xy-(-1)^{|x||y|}yx(x,y\in gl(m,n))$ is a Lie superalgebra. And we usually call it the general linear Lie superalgebra.

2.2. Lie Superalgebras $A(M-1,N-1)$

Now we present some basics on Lie superalgebras $A(M-1,N-1)$.

In fact, $A(M-1,N-1)=sl(M,N)=\{a\in gl(M,N)\mid \mathrm{str}(a)=0\}$, $M\neq N, M, N\geq 1$.

Remark 3. $sl(m,n)$ is a subsuperalgebra of $gl(m,n)$, which is ordinarily named the special linear Lie superalgebra.

Following [6], one denotes $\mathfrak{h}$ as the Cartan subalgebra of the Lie superalgebra $A(M-1,N-1)$. With respect to $\mathfrak{h}$, the root system $\Delta=\Delta_{\bar{0}}\cup\Delta_{\bar{1}}$ for $A(M-1,N-1)$ is as follows:

$$\Delta_{\bar{0}}=\{\varepsilon_i-\varepsilon_j\mid 1\leq i\neq j\leq M\}\cup\{\delta_k-\delta_l\mid 1\leq k\neq l\leq N\},$$
$$\Delta_{\bar{1}}=\{\pm(\varepsilon_i-\delta_k)\mid 1\leq i\leq M, 1\leq k\leq N\}.$$

Next, we present an important concept in this paper.

Definition 3. (see [8]) A Lie superalgebra L over $\mathbb{F}$ is graded by the root system Δ of $A(M-1,N-1)$ if

(i) L includes $A(M-1,N-1)$;

(ii) L has a root space decomposition $L=\bigoplus_{\mu\in\Delta\cup\{0\}}L_\mu$ relative to a split Cartan subalgebra $\mathfrak{h}$ of $A(M-1,N-1)_{\bar{0}}$, where $L_\mu=\{x\in L\,[h,x]=\mu(h)x\text{ for all }h\in\mathfrak{h}\}$ for $\mu\in\Delta\cup\{0\}$;

(iii) $L_0=\sum_{\mu\in\Delta}[L_\mu,L_{-\mu}]$.

2.3. The Matrix Lie Superalgebras with the Quantum Tori

From [17], we are aware that quantum tori related to q $(0\neq q\in\mathbb{C})$ is the unital associative $\mathbb{C}$ -algebra $\mathbb{C}_q[x^\pm,y^\pm]$ (or, simply as $\mathbb{C}_q$) with the generators $x^\pm, y^\pm$ given by

$$xx^{-1}=yy^{-1}=x^{-1}x=y^{-1}y=1,\ yx=qxy.$$

Then

$$x^m y^n x^s y^t = q^{ns} x^{m+s} y^{n+t},$$

and

$$\mathbb{C}_q = \sum_{m,n\in\mathbb{Z}} \oplus \mathbb{C} x^m y^n.$$

Set

$$\Lambda(q) = \{n \in \mathbb{Z} \mid q^n = 1\}.$$

From [18], we know that elements $x^m y^n$ for $m \notin \Lambda(q)$ or $n \notin \Lambda(q)$ form the bases of $[\mathbb{C}_q, \mathbb{C}_q]$.

We have a Lie superalgebra $gl(M,N)(\mathbb{C}_q)$ $(M,N \in \mathbb{N}^*)$ of $(M+N)$ by $(M+N)$ matrices with entries from $\mathbb{C}_q$. Then following the central extensions of general linear Lie algebras $gl_r(\mathbb{C}_q)$ by professor Gao in [4], we form a central extension of the Lie superalgebra $gl(M,N)(\mathbb{C}_q)$,

$$\widehat{gl(M,N)}\left(\mathbb{C}_q\right) = gl(M,N)\left(\mathbb{C}_q\right) \oplus \left(\sum_{n\in\Lambda(q)} \oplus \mathbb{C}c(n)\right) \oplus \mathbb{C}c_y$$

with the Lie superbracket

$$\begin{aligned}
&[A(x^m y^n), B(x^s y^t)] \\
&= A(x^m y^n)B(x^s y^t) - (-1)^{\deg A \deg B} B(x^s y^t) A(x^m y^n) \\
&\quad + mq^{ns}\,\mathrm{str}(AB)\delta_{m+s,0}\delta_{\overline{n+t},\bar{0}}c(n+t) \\
&\quad + nq^{ns}\,\mathrm{str}(AB)\delta_{m+s,0}\delta_{n+t,0}c_y,
\end{aligned} \tag{1}$$

where $m,n,s,t \in \mathbb{Z}$; $A,B \in gl(M,N)_\alpha, \alpha \in \mathbb{Z}_2$; $u \in \Lambda(q)$ in $c(u)$, $\bar{p} \in \mathbb{Z} - \Lambda(q)$ for $p \in \mathbb{Z}$, 'str' represents the supertrace of $gl(M,N)$ and c_y are central elements of $\widehat{gl(M,N)}\left(\mathbb{C}_q\right)$.

3. A(M-1,N-1)-Graded Lie Superalgebras

In this section, we first present the construction of $A(M-1,N-1)$-graded Lie superalgebras with the quantum tori as the coordinate algebra. And then we construct the representation of $A(M-1,N-1)$-graded Lie superalgebras by using bosons and fermions.

3.1. The Construction of A(M-1,N-1)-Graded Lie Superalgebras

In this subsection, we study the structure of $A(M-1,N-1)$-graded Lie superalgebras with coordinate algebra as quantum tori.

We set $\mathcal{G} = sl(M,N)(\mathbb{C}_q)$, i.e.

$$\mathcal{G} = [gl(M,N)(\mathbb{C}_q), gl(M,N)(\mathbb{C}_q)] = \{A \in gl(M,N)(\mathbb{C}_q) \mid \mathrm{str}(A) \equiv 0 \bmod [\mathbb{C}_q, \mathbb{C}_q]\}.$$

Remark 4. $\mathcal{G}$ is an $A(M-1,N-1)$-graded Lie superalgebra (since $\mathbb{C}_q$ is unital) with $\mathbb{C}_q$ as a coordinate superalgebra.

Then let $\mathcal{G}_\alpha = \{x \in \mathcal{G} [h,x] = \alpha(h)x, \text{ for all } h \in \mathcal{H}\}$.

We have the following root space decomposition:

$$\mathcal{G} = \mathcal{G}_0 \oplus \sum_{i\neq j}(\mathcal{G}_{\varepsilon_i-\varepsilon_j} \oplus \mathcal{G}_{\delta_i-\delta_j}) \oplus \sum_{i,j}(\mathcal{G}_{\varepsilon_i-\delta_j} \oplus \mathcal{G}_{-\varepsilon_i+\delta_j}),$$

here

$$\begin{aligned}
\mathcal{G}_{\varepsilon_i-\varepsilon_j} &= \mathrm{span}_{\mathbb{C}}\{\tilde{f}_{ij}(m,n) := x^m y^n e_{ij}\}, i \neq j, \\
\mathcal{G}_{\delta_k-\delta_l} &= \mathrm{span}_{\mathbb{C}}\{\tilde{g}_{kl}(m,n) := x^m y^n e_{M+k,M+l}\}, k \neq l, \\
\mathcal{G}_{\varepsilon_i-\delta_k} &= \mathrm{span}_{\mathbb{C}}\{\tilde{F}_{ik}(m,n) := x^m y^n e_{i,M+k}\}, \\
\mathcal{G}_{-\varepsilon_i+\delta_k} &= \mathrm{span}_{\mathbb{C}}\{\tilde{G}_{ik}(m,n) := x^m y^n e_{M+k,i}\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}_0 = {} & \mathrm{span}_{\mathbb{C}}\{\tilde{f}_{ii}(m,n) - \tilde{f}_{i+1,i+1}(m,n) \mid 1 \le i \le M-1\} \\
& \oplus \mathrm{span}_{\mathbb{C}}\{\tilde{g}_{kk}(m,n) - \tilde{g}_{k+1,k+1}(m,n) \mid 1 \le k \le N-1\} \\
& \oplus \mathrm{span}_{\mathbb{C}}\{\tilde{f}_{MM}(m,n) + \tilde{g}_{M+1,M+1}(m,n)\},
\end{aligned}$$

where $1 \le i,j \le M, 1 \le k,l \le N$, and $m,n \in \mathbb{Z}$.

Then one gets a central extension of $\mathcal{G}$

$$\hat{\mathcal{G}} = \mathcal{G} \oplus \left(\sum_{n \in \Lambda(q)} \oplus \mathbb{C}c(n) \right) \oplus \mathbb{C}c_y$$

with the Lie superbracket as (1).

Remark 5. $\hat{\mathcal{G}}$ is an $A(M-1, N-1)$ -graded Lie superalgebra.

Now we give the multiplication of Lie superalgebra $\hat{\mathcal{G}}$ specifically.

Proposition 1.

$$[\tilde{f}_{ij}(m,n), \tilde{f}_{kl}(p,s)] = \delta_{jk} q^{np} \tilde{f}_{il}(m+p, n+s) - \delta_{il} q^{ms} \tilde{f}_{kj}(m+p, n+s) + mq^{np} \delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+s},\bar{0}} c(n+s) + nq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{n+s,0}c_y;$$

$$[\tilde{f}_{ij}(m,n), \tilde{g}_{kl}(p,s)] = 0;$$

$$[\tilde{f}_{ij}(m,n), \tilde{F}_{kl}(p,s)] = \delta_{jk} q^{np} \tilde{F}_{il}(m+p, n+s);$$

$$[\tilde{f}_{ij}(m,n), \tilde{G}_{kl}(p,s)] = -\delta_{ik} q^{ms} \tilde{G}_{jl}(m+p, n+s);$$

$$[\tilde{g}_{ij}(m,n), \tilde{g}_{kl}(p,s)] = \delta_{jk} q^{np} \tilde{g}_{il}(m+p, n+s) - \delta_{il} q^{ms} \tilde{g}_{kj}(m+p, n+s) - mq^{np} \delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+s},\bar{0}} c(n+s) - nq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{n+s,0}c_y;$$

$$[\tilde{g}_{ij}(m,n), \tilde{F}_{kl}(p,s)] = -\delta_{il} q^{ms} \tilde{F}_{kj}(m+p, n+s);$$

$$[\tilde{g}_{ij}(m,n), \tilde{G}_{kl}(p,s)] = \delta_{jl} q^{np} \tilde{G}_{ki}(m+p, n+s);$$

$$[\tilde{F}_{ij}(m,n), \tilde{F}_{kl}(p,s)] = 0;$$

$$[\tilde{F}_{ij}(m,n), \tilde{G}_{kl}(p,s)] = \delta_{jl} q^{np} \tilde{f}_{ik}(m+p, n+s) + \delta_{ik} q^{ms} \tilde{g}_{lj}(m+p, n+s) + mq^{np} \delta_{ik}\delta_{jl}\delta_{m+p,0}\delta_{\overline{n+s},\bar{0}} c(n+s) + nq^{np}\delta_{ik}\delta_{jl}\delta_{m+p,0}\delta_{n+s,0}c_y;$$

$$[\tilde{G}_{ij}(m,n), \tilde{G}_{kl}(p,s)] = 0.$$

3.2. The Representations of A(M-1,N-1)-graded Lie Superalgebras

In this subsection, fermionic-bosonic operators are used to achieve a family of $A(M-1, N-1)$ -graded Lie superalgebras whose coordinate algebra is a quantum tori.

Now we follow the method in [10, 19] to construct representations for the Lie superalgebras $\hat{\mathcal{G}}$ which are given in sec. 3.1.

Let $\mathfrak{R}$ be an arbitrary associative algebra. For $\tau = \pm 1,$ we define a τ -bracket on $\mathfrak{R}$ as follows:

$$\{x, y\}_\tau = xy + \tau yx, \quad x, y \in \mathfrak{R}.$$

Obviously, we have $\{x, y\}_\tau = \tau\{y, x\}_\tau$ and $[xy, z] = x\{y, z\}_\tau - \tau\{x, z\}_\tau y.$

Let $\mathfrak{b}$ be a unital associative algebra with generators $h_i, h_i^*, 1 \leqslant i \leqslant X \in \mathbb{N}^*$ satisfying

$$\left\{h_i, h_j\right\}_\tau = \left\{h_i^*, h_j^*\right\}_\tau = 0, \quad \left\{h_i, h_j^*\right\}_\tau = \delta_{ij}.$$

Let $\alpha(X, \tau)$ be the associative algebra generated by

$$\left\{ b(s) | \; b \in \bigoplus_{i=1}^{N} \left(\mathbb{C}h_i \oplus \mathbb{C}h_i^* \right), s \in \mathbb{Z} \right\}$$

satisfying

$$\{b(s), d(t)\}_\tau = \{b, d\}_\tau \delta_{s+t,0}.$$

Then we define the normal ordering as in [6]:

$$:b(s)d(t): = \begin{cases} b(s)d(t), & s > t, \\ \frac{1}{2}(b(s)d(t) - \tau d(t)b(s)), & s = t, \\ -\tau d(t)b(s), & s < t, \end{cases}$$
$$= -\tau : d(t)b(s):$$

for $s, t \in \mathbb{Z},$ and $b, d \in \mathfrak{b}.$ Set

$$\varphi(s) = \begin{cases} 1, s > 0, \\ \frac{1}{2}, s = 0, \\ 0, s < 0. \end{cases}$$

Then we have

$$:h_i(s)h_j(t) := h_i(s)h_j(t) = -\tau h_j(t)h_i(s), \quad (2)$$

$$:h_i^*(s)h_j^*(t) := h_i^*(s)h_j^*(t) = -\tau h_j^*(t)h_i^*(s), \quad (3)$$

$$h_i(s)h_j^*(t) =: h_i(s)h_j^*(t) : +\delta_{ij}\delta_{s+t,0}\varphi(s-t), \quad (4)$$

$$h_j^*(t)h_i(s) =: h_j^*(t)h_i(s) : +\tau\delta_{ij}\delta_{s+t,0}\varphi(t-s). \quad (5)$$

Next, we denote the generators of $\alpha(M, +1)$ by $a_i(s), a_j^*(t)$ and the ones of $\alpha(N, -1)$ by $e_i(s), e_j^*(t)$. By a direct computation, we have

Proposition 2. The subspaces of both Clifford algebra

$\alpha(M,+1)$ and Weyl algebra $\alpha(N,-1)$ consisted of fermionic and bosonic quadratic operators, respectively, are closed under the Lie bracket $[\cdot,\cdot]_-$.

Proposition 3. The subspace of the tensor product algebra $\alpha(M,+1)\otimes\alpha(N,-1)$ composed of fermionic-bosonic quadratic operators is closed under the Jordan bracket $[\cdot,\cdot]_+$. Furthermore, set $b(s)\otimes 1=b(s), 1\otimes d(t)=d(t),$ then $b(s)\otimes d(t)=b(s)d(t)=d(t)b(s)$.

As in [10, 19], let $\alpha(X,\tau)^+$ be the subalgebra generated by $h_j^*(s), h_i(t)$ and $h_k^*(0)$ for $s,t>0$ and $1\leqslant i,j,k\leqslant X$. Let $\alpha(X,\tau)^-$ be the subalgebra generated by $h_j^*(s), h_i(t),$ and $h_k(0)$ for $s,t<0$ and $1\leqslant i,j,k\leqslant X$. Let $V(X,\tau)$ be a simple $\alpha(X,\tau)$ -module containing an element v_0^τ, satisfying

$$\alpha(X,\tau)^+ v_0^\tau = 0$$

and

$$V(X,\tau)=\alpha(X,\tau)^- v_0^\tau.$$

The rest of normal orderings are defined as follows:

$$:a_i(s)\, e_j(t) := a_i(s)\, e_j(t), \tag{6}$$

$$:a_i(s)e_j^*(t) := a_i(s)e_j^*(t), \tag{7}$$

$$:a_i^*(s)e_j(t) := a_i^*(s)e_j(t), \tag{8}$$

$$:a_i^*(s)e_j^*(t) := a_i^*(s)e_j^*(t). \tag{9}$$

Clearly, the $\alpha(M,+1)\otimes\alpha(N,-1)$ -module $V(M,N):=V(M,+1)\otimes V(N,-1)=\alpha(M,+1)\otimes\alpha(N,-1)v_0^{+1}\otimes v_0^{-1}$ is simple.

Let

$$f_{ij}(m,n)=\sum_{s\in\mathbb{Z}} q^{-ns} :a_i(m-s)a_j^*(s): - \begin{cases} 0, & n\in\Lambda(q), \\ \frac{1}{2}\delta_{ij}\delta_{m,0}\frac{q^n+1}{q^n-1}, & n\in\mathbb{Z}\setminus\Lambda(q), \end{cases}$$

$$g_{ij}(m,n)=\sum_{s\in\mathbb{Z}} q^{-ns} :e_i(m-s)e_j^*(s): + \begin{cases} 0, & n\in\Lambda(q), \\ \frac{1}{2}\delta_{ij}\delta_{m,0}\frac{q^n+1}{q^n-1}, & n\in\mathbb{Z}\setminus\Lambda(q), \end{cases}$$

$$F_{ij}(m,n)=\sum_{s\in\mathbb{Z}} q^{-ns} :a_i(m-s)e_j(s):,$$

$$G_{ij}(m,n)=\sum_{s\in\mathbb{Z}} q^{-ns} :a_i^*(s)e_j^*(m-s):.$$

To calculate the commutator relations of these operators, we give the following lemma.

Lemma 1.

$$[a_i(m)a_j^*(n), a_k(p)a_l^*(s)]_- = \delta_{jk}\delta_{n+p,0}a_i(m)a_l^*(s) - \delta_{il}\delta_{m+s,0}a_k(p)a_j^*(n), \tag{10}$$

$$[a_i(m)a_j^*(n), e_k^*(p)e_l(s)]_- = 0, \tag{11}$$

$$[a_i(m)a_j^*(n), a_k^*(p)e_l^*(s)]_- = -\delta_{ik}\delta_{n+p,0}a_j^*(n)e_l^*(s), \tag{12}$$

$$[a_i(m)a_j^*(n), a_k(p)e_l(s)]_- = \delta_{jk}\delta_{n+p,0}a_i(m)e_l(s), \tag{13}$$

$$[e_i^*(m)e_j(n), e_k^*(p)e_l(s)]_- = \delta_{jk}\delta_{n+p,0}e_i^*(m)e_l(s) - \delta_{il}\delta_{m+s,0}e_k^*(p)e_j(n), \tag{14}$$

$$[e_i^*(m)e_j(n), a_k(p)e_l(s)]_- = -\delta_{il}\delta_{m+s,0}a_k(p)e_j(n), \tag{15}$$

$$[e_i^*(m)e_j(n), a_k^*(p)e_l^*(s)]_- = \delta_{jl}\delta_{n+s,0}a_k^*(p)e_i^*(m), \tag{16}$$

$$[a_i(m)e_j(n), a_k(p)e_l(s)]_+ = 0, \tag{17}$$

$$[a_i(m)e_j(n), a_k^*(p)e_l^*(s)]_+ = \delta_{ik}\delta_{m+p,0}e_l^*(s)e_j(n) + \delta_{jl}\delta_{n+s,0}a_i(m)a_k^*(p), \tag{18}$$

$$[a_i^*(m)e_j^*(n), a_k^*(p)e_l^*(s)]_+ = 0. \tag{19}$$

Later, if $n \in \Lambda(q)$, let $\dfrac{q^{mn}-1}{q^n-1} = m$ to avoid ambiguity.

Proposition 4.

$$[f_{ij}(m,n), f_{kl}(p,s)] = \delta_{jk}q^{np}f_{il}(m+p,n+s) - \delta_{il}q^{ms}f_{kj}(m+p,n+s) + mq_{np}\delta_{il}\delta_{jk}\delta_{m+p,0}\delta_{\overline{n+s},\bar{0}}, \tag{20}$$

$$[f_{ij}(m,n), g_{kl}(p,s)] = 0, \tag{21}$$

$$[f_{ij}(m,n), F_{kl}(p,s)] = \delta_{jk}q^{np}F_{il}(m+p,n+s), \tag{22}$$

$$[f_{ij}(m,n), G_{kl}(p,s)] = -\delta_{ik}q^{ms}G_{jl}(m+p,n+s), \tag{23}$$

$$[g_{ij}(m,n), g_{kl}(p,s)] = \delta_{jk}q^{np}g_{il}(m+p,n+s) - \delta_{il}q^{ms}g_{kj}(m+p,n+s) - mq_{np}\delta_{il}\delta_{jk}\delta_{m+p,0}\delta_{\overline{n+s},\bar{0}}, \tag{24}$$

$$[g_{ij}(m,n), F_{kl}(p,s)] = -\delta_{il}q^{ms}F_{kj}(m+p,n+s), \tag{25}$$

$$[g_{ij}(m,n), G_{kl}(p,s)] = \delta_{jl}q^{np}G_{ki}(m+p,n+s), \tag{26}$$

$$[F_{ij}(m,n), F_{kl}(p,s)] = 0, \tag{27}$$

$$[F_{ij}(m,n), G_{kl}(p,s)] = \delta_{jl}q^{np}f_{ik}(m+p,n+s) + \delta_{ik}q^{ms}g_{lj}(m+p,n+s) + mq_{np}\delta_{ik}\delta_{jl}\delta_{m+p,0}\delta_{\overline{n+s},\bar{0}}, \tag{28}$$

$$[G_{ij}(m,n), G_{kl}(p,s)] = 0. \tag{29}$$

Proof: Notice that constants have no effect for Lie bracket.
From (4) and (10), we have

$$\begin{aligned}
&[f_{ij}(m,n), f_{kl}(p,s)] \\
&= \sum_{t,r\in\mathbb{Z}} q^{-nt-sr}[a_i(m-t)a_j^*(t), a_k(p-r)a_l^*(t)] \\
&= \sum_{t,r\in\mathbb{Z}} q^{-nt-sr}\left(\delta_{jk}\delta_{p+t-r,0}a_i(m-t)a_l^*(r) - \delta_{il}\delta_{m+r-t,0}a_k(p-r)a_j^*(t)\right) \\
&= \sum_{t,r\in\mathbb{Z}} q^{-nt-sr}\delta_{jk}\delta_{p+t-r,0}\left(:a_i(m-t)a_l^*(r): + \delta_{il}\delta_{m+r-t,0}\theta(m-t-r)\right) - \sum_{t,r\in\mathbb{Z}} q^{-nt-sr}\delta_{il}\delta_{m+r-t,0}\left(:a_k(p-r)a_j^*(t): + \delta_{jk}\delta_{p+t-r,0}\theta(p-r-t)\right) \\
&= \delta_{jk}q^{np}\sum_{r\in\mathbb{Z}} q^{-r(n+s)} :a_i(m+p-r)a_l^*(r): - \delta_{il}q^{ms}\sum_{t\in\mathbb{Z}} q^{-t(n+s)} :a_k(m+p-t)a_j^*(t): + \delta_{il}\delta_{jk}\delta_{m+p,0}q^{np}\sum_{r\in\mathbb{Z}} q^{-r(n+s)}\left(\theta(-2r) - \theta(-2m-2r)\right).
\end{aligned}$$

Since

$$\sum_{r\in\mathbb{Z}} q^{-r(n+s)}(\theta(-2r) - \theta(-2m-2r)) = \begin{cases} 0, & m = 0, \\ \dfrac{1}{2}(1+q^{m(n+s)}) + \displaystyle\sum_{r=-m+1}^{-1} q^{-r(n+s)}, m > 0, = \dfrac{1}{2}(q^{(n+s)}+1)\dfrac{q^{m(n+s)}-1}{q^{(n+s)}-1}, & \\ -\dfrac{1}{2}(1+q^{m(n+s)}) - \displaystyle\sum_{r=1}^{-m-1} q^{-r(n+s)}, m < 0, & \end{cases} \tag{30}$$

two cases will be discussed as follows.

Case 1: If $n+s\in\Lambda(q)$, then

$$[f_{ij}(m,n),f_{kl}(p,s)]=\delta_{jk}q^{np}\sum_{r\in\mathbb{Z}}q^{-r(n+s)}:a_i(m+p-r)a_l^*(r):-\delta_{il}q^{ms}\sum_{t\in\mathbb{Z}}q^{-t(n+s)}:a_k(m+p-t)a_j^*(t):+mq^{np}\delta_{il}\delta_{jk}\delta_{m+p,0}.$$

Case 2: If $n+s\in\mathbb{Z}\setminus\Lambda(q)$, then

$$\begin{aligned}&[f_{ij}(m,n),f_{kl}(p,s)]\\&=\delta_{jk}q^{np}\sum_{r\in\mathbb{Z}}q^{-r(n+s)}:a_i(m+p-r)a_l^*(r):-\delta_{il}q^{ms}\sum_{t\in\mathbb{Z}}q^{-t(n+s)}:a_k(m+p-t)a_j^*(t):+\frac{1}{2}\delta_{il}\delta_{jk}\delta_{m+p,0}\frac{q^{(n+s)}+1}{q^{(n+s)}-1}(q^{ms}-q^{np})\\&=\delta_{jk}q^{np}\left(\sum_{r\in\mathbb{Z}}q^{-r(n+s)}:a_i(m+p-r)a_l^*(r):-\frac{1}{2}\delta_{il}\delta_{m+p,0}\frac{q^{(n+s)}+1}{q^{(n+s)}-1}\right)-\delta_{il}q^{ms}\left(\sum_{t\in\mathbb{Z}}q^{-t(n+s)}:a_k(m+p-t)a_j^*(t):-\frac{1}{2}\delta_{jk}\delta_{m+p,0}\frac{q^{(n+s)}+1}{q^{(n+s)}-1}\right).\end{aligned}$$

Hence, one obtains that

$$[f_{ij}(m,n),f_{kl}(p,s)]=\delta_{jk}q^{np}f_{il}(m+p,n+s)-\delta_{il}q^{ms}f_{kj}(m+p,n+s)+mq_{np}\delta_{il}\delta_{jk}\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}.$$

This means that (20) is true.
From (6) and (11), one has

$$\begin{aligned}[f_{ij}(m,n),F_{kl}(p,s)]&=\sum_{t,r\in\mathbb{Z}}q^{-nt-sr}[a_i(m-t)a_j^*(t),a_k(p-r)e_l(r)]=\sum_{t,r\in\mathbb{Z}}q^{-nt-sr}\delta_{jk}\delta_{p+t-r,0}a_i(m-t)e_l(r)\\&=\delta_{jk}q^{np}\sum_{r\in\mathbb{Z}}q^{-r(n+s)}a_i(m+p-r)e_l(r)=\delta_{jk}q^{np}F_{il}(m+p,n+s).\end{aligned}$$

This means that (22) is true.
From (4), (5), (18) and (30), we have

$$\begin{aligned}&[F_{ij}(m,n),G_{kl}(p,s)]\\&=\sum_{t,r\in\mathbb{Z}}q^{-nt-sr}[a_i(m-t)e_j(t),a_k^*(r)e_l^*(p-r)]\\&=\sum_{t,r\in\mathbb{Z}}q^{-nt-sr}\left(\delta_{ik}\delta_{m+r-t,0}e_l^*(p-r)e_j(t)+\delta_{jl}\delta_{p+t-r,0}a_i(m-t)a_k^*(r)\right)\\&=\sum_{t,r\in\mathbb{Z}}q^{-nt-sr}\delta_{ik}\delta_{m+r-t,0}\left(:e_l^*(p-r)e_j(t):-\delta_{jl}\delta_{p+t-r,0}\theta(p-r-t)\right)+\sum_{t,r\in\mathbb{Z}}q^{-nt-sr}\delta_{jl}\delta_{p+t-r,0}\left(:a_i(m-t)a_k^*(r):+\delta_{ik}\delta_{m+r-t,0}\theta(m-t-r)\right)\\&=\delta_{ik}q^{ms}\sum_{t\in\mathbb{Z}}q^{-t(n+s)}:e_l^*(m+p-t)e_j(t):+\delta_{jl}q^{np}\sum_{r\in\mathbb{Z}}q^{-r(n+s)}:a_i(m+p-r)a_k^*(r):+\delta_{ik}\delta_{jl}\delta_{m+p,0}q^{np}\sum_{r\in\mathbb{Z}}q^{-r(n+s)}(\theta(-2r)-\theta(-2m-2r))\\&=\delta_{ik}q^{ms}\sum_{t\in\mathbb{Z}}q^{-t(n+s)}:e_l^*(m+p-t)e_j(t):+\delta_{jl}q^{np}\sum_{r\in\mathbb{Z}}q^{-r(n+s)}:a_i(m+p-r)a_k^*(r):+\delta_{ik}\delta_{jl}\delta_{m+p,0}q^{np}\frac{1}{2}(q^{(n+s)}+1)\frac{q^{m(n+s)}-1}{q^{(n+s)}-1}.\end{aligned}$$

When $n+s\in\Lambda(q)$, we get

$$[F_{ij}(m,n),G_{kl}(p,s)]=\delta_{ik}q^{ms}\sum_{t\in\mathbb{Z}}q^{-t(n+s)}:e_l^*(m+p-t)e_j(t):+\delta_{jl}q^{np}\sum_{r\in\mathbb{Z}}q^{-r(n+s)}:a_i(m+p-r)a_k^*(r):+mq^{np}\delta_{ik}\delta_{jl}\delta_{m+p,0}.$$

When $n+s\in\mathbb{Z}\setminus\Lambda(q)$, we get

$$\begin{aligned}&[F_{ij}(m,n),G_{kl}(p,s)]\\&=\delta_{ik}q^{ms}\sum_{t\in\mathbb{Z}}q^{-t(n+s)}:e_l^*(m+p-t)e_j(t):+\delta_{jl}q^{np}\sum_{r\in\mathbb{Z}}q^{-r(n+s)}:a_i(m+p-r)a_k^*(r):+\frac{1}{2}\delta_{jl}\delta_{ik}\delta_{m+p,0}\frac{q^{(n+s)}+1}{q^{(n+s)}-1}(q^{ms}-q^{np})\\&=\delta_{jl}q^{np}\left(\sum_{r\in\mathbb{Z}}q^{-r(n+s)}:a_i(m+p-r)a_k^*(r):-\frac{1}{2}\delta_{ik}\delta_{m+p,0}\frac{q^{(n+s)}+1}{q^{(n+s)}-1}\right)+\delta_{ik}q^{ms}\left(\sum_{t\in\mathbb{Z}}q^{-t(n+s)}:e_l^*(m+p-t)e_j(t):+\frac{1}{2}\delta_{jl}\delta_{m+p,0}\frac{q^{(n+s)}+1}{q^{(n+s)}-1}\right).\end{aligned}$$

Therefore, one obtains that

$$[F_{ij}(m,n),G_{kl}(p,s)]=\delta_{jl}q^{np}f_{ik}(m+p,n+s)+\delta_{ik}q^{ms}g_{lj}(m+p,n+s)+mq^{np}\delta_{ik}\delta_{jl}\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}.$$

This means that (28) is true.

Finally, the proof of the others can be received similarly from Lemma 1. □

Theorem 1. $V(M,N)$ *is a module of Lie superalgebra* $\hat{\mathcal{G}}$ *subject to relations*

$$\pi(\tilde{f}_{ij}(m,n))=f_{ij}(m,n),\quad \pi(\tilde{g}_{ij}(m,n))=g_{ij}(m,n),$$
$$\pi(\tilde{F}_{ij}(m,n))=F_{ij}(m,n),\quad \pi(\tilde{G}_{ij}(m,n))=G_{ij}(m,n),$$
$$\pi(c(n))=1,\qquad \pi(c_y)=0.$$

4. Conclusion

In this paper, we studied the representation theory of $A(M-1,N-1)$ -graded Lie superalgebras with coordinate algebra as quantum tori. We first gave the structure of $A(M-1,N-1)$ -graded Lie superalgebras concretely, i.e., a set of bases and their multiplication operations. And finally, we constructed the representations by using the tensor product module which are extremely concise and pretty results.

References

[1] Varadarajan V S. (2004). Supersymmetry for mathematicians. New York: American Mathematical Society.

[2] Berman S, Moody R V. (1992). Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy. Inventiones mathematicae, 108 (1): 323-347.

[3] Benkart G, Zelmanov E. (1996). Lie algebras graded by finite root systems and intersection matrix algebras. Inventiones mathematicae, 126 (1): 1-45.

[4] Neher E. (1996). Lie algebras graded by 3-graded root systems and Jordan pairs covered by grids. American Journal of Mathematics. 118 (2): 439-491.

[5] Allison B N, Benkart G, Gao Y. (2000). Central extensions of Lie algebras graded by finite root systems. Mathematische Annalen. 316 (3): 499-527.

[6] Kac V G. (1977). Lie superalgebras. Advances in Math, 26 (1): 8-96.

[7] P Benkart G, Elduque A. (2002). Lie superalgebras graded by the root systems $C(n)$, $D(m,n)$, $D(2,1;\alpha)$, $F(4)$, $G(3)$. Canad Math Bull, 45 (4): 509-524. 0.

[8] Benkart G, Elduque A. (2002). Lie superalgebras graded by the root system $A(m,n)$.

[9] Benkart G, Elduque A. (2003). Lie superalgebras graded by the root system $B(m,n)$. Selecta Mathematica, New Series, 9: 313-360.

[10] Gao Y. (2002). Fermionic and bosonic representations of the extended affine Lie algebra $gl_N(C_q)$ Canadian Mathematical Bulletin, 45 (4): 623-633.

[11] Lau M. (2005). Bosonic and fermionic representations of Lie algebra central extensions. Advances in Mathematics, 194 (2): 225-245.

[12] Chen H, Gao Y. (2007). BC_N-graded Lie algebras arising from fermionic representations. J Algebra, 308 (2): 1740-1752.

[13] Chen H, Gao Y. (2006). $B(0,N)$-graded Lie superalgebras coordinatized by quantum tori. Sci China Ser A, 49: 545-566 2.

[14] Cheng J. (2016). Generalized $B(m,n)$, $C(n)$, $D(m,n)$-graded Lie superalgebras arising from fermionic-bosonic representations. Frontiers of Mathematics in China, 11 (6): 1451-1470.

[15] Cheng J, Gao Y. Generalized $P(N)$-graded Lie superalgebras. submitted.

[16] Cheng J. (2016). $Q(N)$-graded Lie superalgebras arising from fermionic-bosonic representations. Pacific Journal of Mathematics, 283 (1): 63-74.

[17] Manin Y I. (1991). Topics in noncommutative geometry. M. B. Porter Lectures. Princeton: Princeton University Press.

[18] Berman S, Gao Y, Krylyuk Y. (1996). Quantum tori and the structure of elliptic quasi-simple Lie algebras. J Funct Anal, 135 (2): 339-389.

[19] Feingold A J, Frenkel I B. (1985). Classical affine algebras. Adv Math, 56 (2): 117-172.