

A Family of Global Attractors for the Higher-order Kirchhoff-type Equations and Its Dimension Estimation

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Abstract: In this paper, we study the long-time behavior of solutions for a class of initial boundary value problems of higher order Kirchhoff-type equations, and make appropriate assumptions about the Kirchhoff stress term. We use the uniform prior estimation and Galerkin method to prove the existence and uniqueness of the solution of the equation, when the order m and the order q meet certain conditions. Then, we use the prior estimation to get the bounded absorption set, it is further proved that using the Rellich-Kondrachov compact embedding theorem, the solution semigroup generated by the equation has a family of global attractor. Then the equation is linearized and rewritten into a first-order variational equation, and it is proved that the solution semigroup is Frechet differentiable. Finally, it proves that the Hausdorff dimension and Fractal dimension of a family of global attractors are finite.

Keywords: Kirchhoff-Type Equation, Prior Estimation, Galerkin Method, A family of Global Attractors, Hausdorff Dimension, Fractal Dimension

1. Introduction

In this paper, we consider the initial-boundary value problem of the following higher-order nonlinear Kirchhoff-type equations

$$u_{tt} + a(t)(-\Delta)^m u_t + \Delta^{2m} u + N(\|D^m u(t)\|_q^q)(-\Delta)^m u = I(x), \quad (1)$$

$$u(x, t) = 0, \quad \frac{\partial^i u}{\partial \nu^i} = 0, \quad i = 1, 2, \dots, 2m-1, x \in \partial\Omega, \quad t > 0, \quad (2)$$

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega \subset R^n. \quad (3)$$

Where Ω is a bounded domain in $R^n (n \geq 1)$ with smooth boundary $\partial\Omega$, $m \in N^+$, $\partial\Omega$ is the homogeneous Dirichlet boundary of Ω , ν is an outer normal vector of $\partial\Omega$, $a(t)$ is a general function of t , and $I(x)$ is an external force term. The assumptions about $a(t)$ and $N(\|D^m u(t)\|_q^q)$ will be given later.

In 1883, Gustav Robert Kirchhoff [1], a German physicist, first introduced the following equation when he studied the transverse vibration of stretched strings

$$\rho h \frac{\partial^2 u}{\partial t^2} - \left\{ P_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial t} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < L, \quad t \geq 0,$$

Where h is the cross-sectional area of the string, E is young coefficient, P_0 is the initial axial tension, L is the length of the string, ρ is the mass density of the string, and $u = u(x, t)$ is the lateral displacement in the spatial axial coordinate. Over the past a hundred years, with the development of science and technology, Kirchhoff equation has been paid more and more attention by scholars. The application field of Kirchhoff equation is also expanding, and its expression is constantly extended. A series of mathematical theories and research results have been obtained, such as the existence and uniqueness of global solution, the decay of solution, the existence of random attractor, and the index Numerical attractor, global attractor and its dimension estimation, inertial manifold, etc [2-6].

When $m = 1, q = 2$, the stress term is $(1 + \|\nabla u(t)\|_2^2)$, and a nonlinear nonlocal source term is added to equation (1), Mitsuhiro Nakao [7] studied the existence of attractors and some absorption properties in the local sense for this class of Kirchhoff type quasilinear wave equation with standard dissipation term u_t . Then, Zaiyun Zhang [8] and others studied the initial boundary value problem of nonlinear dissipative Kirchhoff equation when $m = 0, q = 2, a(t) = \beta$, $N(s) = M(\|u(t)\|^2)$, the initial boundary value problem of nonlinear dissipative Kirchhoff equation is as follows

$$u_{tt} + M(\|u\|^2)\Delta u + \beta u_t + g(u) = f(x),$$

In Hilbert spaces $H_0^1(\Omega) \times L^2(\Omega)$ and $D(A) \times H_0^1(\Omega)$, they estimate the dimensions of global attractors by using rescaling technique and linearized variational method. The upper bounds of Hausdorff dimension and Fractal dimension are obtained.

Furthermore, when $m = 1, q = 2$, stress term $N(s) = \sigma(\|Du\|^2)$ and strong dissipation term $a(t) = \sigma(\|Du\|^2)$, chueshov Igor [9] studied the well posedness and long-time behavior of the solution of the initial boundary value problem of the equation

$$u_{tt} - \sigma(\|Du\|^2)\Delta u_t - \varphi(\|Du\|^2)\Delta u + g(u) = h(x),$$

By assuming the Kirchhoff term, he proved the existence and uniqueness of weak solution and the existence of a finite dimensional global attractor in the natural energy space with partial strong topology, and further proved that the attractor is strong under non supercritical conditions.

Recently, on the basis of chueshov Igor [9], Guoguang Lin [10] and others studied the long-term behavior of the initial boundary value problem for a class of nonlinear strongly damped higher order Kirchhoff type equation with $m = 1, q = 2, a(t) = 1$ and $N(\|D^m u(t)\|_2^2) = (\|D^m u(t)\|_2^2)^q$

$$u_{tt} + (-\Delta)^m u_t + \|\nabla^m u\|^{2q} (-\Delta)^m u + g(u) = f(x),$$

They obtain the existence and uniqueness of the solution and the global attractor, and consider the dimension of the global attractor and the upper bound estimation of the dimension. For more related research results on the Kirchhoff equation, please refer to [11-15].

In this paper, based on the long-time behavior of solutions of some nonlinear Kirchhoff type equations with initial boundary value problems, a class of higher order Kirchhoff type equations with the highest order term $\Delta^{2m} u$ ($m \in \mathbb{N}^+$) is studied. Because of the uncertainty of m in Kirchhoff stress term $N(\|D^m u(t)\|_q^q)$ and q in Banach space $L^q(\Omega)$, there will be a bottleneck when using uniform prior estimation and Galerkin finite element method to prove the existence and

uniqueness of the global solution of the equation, so it is impossible to continue the follow-up work. We get a relation between m and q in Banach space $L^q(\Omega)$ by the theory of Sobolev space. Therefore, we overcome this problem successfully and get more extensive research methods and theoretical results.

2. Preliminaries

For brevity, we used the follow abbreviation:

$H = L^2(\Omega)$, $D = \nabla$, $\|\cdot\|_{L^2} = \|\cdot\|$, $H_0^m(\Omega) = H^m(\Omega) \cap H_0^1(\Omega)$, $H_0^{2m+k}(\Omega) = H^{2m+k}(\Omega) \cap H_0^1(\Omega)$, $E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega)$, Let A_k be a family of global attractors from E_0 to E_k , and B_{0k} be a bounded absorbing set in E_k , with $k = 0, 1, 2, 3, \dots, 2m$ and $C_i (i = 1, 2, 3, \dots)$ denotes positive constant. The notation $(\cdot, \cdot), \|\cdot\|$ for the H inner product and norm, that is $(u, v) = \int_{\Omega} u(x)v(x)dx$, $(u, u) = \|u\|^2$.

Now we state some assumptions and result. Suppose the functions $a(t)$ and $N(\|D^m u(t)\|_q^q)$ in the equation satisfy the following conditions.

(A1) $a(t) \in C^2[0, \infty)$ and $\forall t \geq 0$, such that

$$2 + \varepsilon \leq \mu_0 \leq a(t) \leq \mu_1, \mu = \begin{cases} \mu_0 & \frac{d}{dt} \|D^{m+k} u\|^2 \geq 0 \\ \mu_1 & \frac{d}{dt} \|D^{m+k} u\|^2 < 0 \end{cases}$$

(A2) $N(s) \in C^2[0, \infty)$ and $\forall s \geq 0$, such that

$$1 + \varepsilon \leq \delta_0 \leq N(s) \leq \delta_1, \delta = \begin{cases} \delta_0 & \frac{d}{dt} \|D^{m+k} u\|^2 \geq 0 \\ \delta_1 & \frac{d}{dt} \|D^{m+k} u\|^2 < 0 \end{cases}$$

Where

$$0 < \varepsilon \leq \min \left\{ \frac{\sqrt{1+2\mu_1\lambda_1^m}-1}{2}, \frac{\delta}{\mu}, \frac{2\delta_0}{\lambda_1^{-m}+2\mu_0}, \frac{\delta_1}{\mu_0}, \frac{\sqrt{1+\mu_0\lambda_1^m}-1}{2}, \frac{2\delta_0}{\mu_1+\lambda_1^{-m}} \right\}$$

$$(A3) \begin{cases} \frac{2n}{n+2m} \leq q \leq \frac{2n}{n-2m}, & n > 2m; \\ \frac{2n}{n+2m} \leq q < \infty, & n \leq 2m. \end{cases}$$

3. The Existence of a Family of Global Attractors

Lemma 1 Suppose that the (A1)–(A2) are satisfied, and $I(x) \in H$, $(u_0, u_1) \in E_k$. Then the initial boundary value problem (1)–(3) has a global solution (u, v) satisfies

$$u \in L^\infty(0, +\infty; H_0^{2m+k}(\Omega)),$$

$$v \in L^\infty(0, +\infty; H_0^k(\Omega)) \cap L^2(0, T; H_0^{m+k}(\Omega)), \text{ and}$$

$$\|(u, v)\|_{E_k}^2 = \|D^{2m+k}u\|^2 + \|D^k v\|^2 \leq y_1(0)e^{-\gamma_1 t} + \frac{C_1}{\gamma_1}(1 - e^{-\gamma_1 t}),$$

$$\mu_0 \int_0^T \|D^{m+k}v\|^2 dt \leq y_1(0) + \int_0^T C_1 dt,$$

$$\text{where } v = u_t + \varepsilon u, \quad \gamma_1 = \min \left\{ k_1, \frac{k_2}{\delta}, 2\varepsilon \right\},$$

$$y_1(0) = \|D^k v_0\|^2 + \|D^{2m+k}u_0\|^2 + \delta \|D^{m+k}u_0\|^2.$$

Thus, there exists a non-negative constant R_k and $t_k = t_k(\Omega) > 0$, such that

$$\|(u, v)\|_{E_k}^2 = \|D^{2m+k}u\|^2 + \|D^k v\|^2 \leq R_k^2 \quad (t > t_k).$$

Proof. Let $(-\Delta)^k v = (-\Delta)^k u_t + \varepsilon(-\Delta)^k u$, by taking the inner product in H of Equation (1) with $(-\Delta)^k v$, we obtain

$$(u_{tt} + a(t)(-\Delta)^m u_t + \Delta^{2m}u + N(\|D^m u(t)\|_q^q)(-\Delta)^m u, (-\Delta)^k v) = (I(x), (-\Delta)^k v). \quad (4)$$

By using Hölder Inequality, Young inequality and Poincaré inequality, the following are obtained by dealing with the following items in formula (4)

$$(u_{tt}, (-\Delta)^k v) \geq \frac{1}{2} \frac{d}{dt} \|D^k v\|^2 - \frac{2\varepsilon + \varepsilon^2}{2} \|D^k v\|^2 - \frac{\varepsilon^2 \lambda_1^{-m}}{2} \|D^{m+k}u\|^2. \quad (5)$$

By assuming (A1) and using Young inequality and Poincaré inequality to deal with the strong damping term, we can obtain

$$(a(t)(-\Delta)^m u_t, (-\Delta)^k v) \geq \frac{\mu_0}{2} \|D^{m+k}v\|^2 - \frac{\varepsilon^2 \mu_1}{2} \|D^{m+k}u\|^2, \quad (6)$$

$$(\Delta^{2m}u, (-\Delta)^k v) = (\Delta^{2m}u, (-\Delta)^k (u_t + \varepsilon u)) = \frac{1}{2} \frac{d}{dt} \|D^{2m+k}u\|^2 + \varepsilon \|D^{2m+k}u\|^2 \quad (7)$$

It can be obtained from hypothesis (A2)

$$(N(\|D^m u(t)\|_q^q)(-\Delta)^m u, (-\Delta)^k v) \geq \frac{\delta}{2} \frac{d}{dt} \|D^{m+k}u\|^2 + \varepsilon \delta_0 \|D^{m+k}u\|^2. \quad (8)$$

Using Schwarz inequality and Young inequality to deal with the external force term, we obtain

$$(I(x), (-\Delta)^k v) = (D^k I(x), D^k v) \leq \frac{1}{2\varepsilon^2} \|D^k I(x)\|^2 + \frac{\varepsilon^2}{2} \|D^k v\|^2 \leq C_1 + \frac{\varepsilon^2}{2} \|D^k v\|^2. \quad (9)$$

Substituting (5) - (9) into (4), we receive

$$\begin{aligned} & \frac{d}{dt} (\|D^k v\|^2 + \|D^{2m+k}u\|^2 + \delta \|D^{m+k}u\|^2) + \left(\frac{\mu_0 \lambda_1^m}{2} - 2\varepsilon - 2\varepsilon^2 \right) \|D^k v\|^2 + 2\varepsilon \|D^{2m+k}u\|^2 \\ & + \delta \left(\frac{2\varepsilon \delta_0 - \varepsilon^2 \lambda_1^{-m} - \varepsilon^2 \mu_1}{\delta} \right) \|D^{m+k}u\|^2 + \mu_0 \|D^{m+k}v\|^2 \leq C_1. \end{aligned} \quad (10)$$

According to hypothesis (A1), we have

$$k_1 = \frac{\mu_0 \lambda_1^m}{2} - 2\varepsilon - 2\varepsilon^2 \geq 0, \quad k_2 = 2\varepsilon \delta_0 - \varepsilon^2 \lambda_1^{-m} - \varepsilon^2 \mu_1 \geq 0.$$

where

$$y_1(t) = \|D^k v\|^2 + \|D^{2m+k}u\|^2 + \delta \|D^{m+k}u\|^2. \quad (12)$$

Let $\gamma_1 = \min \left\{ k_1, \frac{k_2}{\delta}, 2\varepsilon \right\}$, then we get

By using Gronwall inequality, we obtain

$$y_1(t) \leq y_1(0)e^{-\gamma_1 t} + \frac{C_1}{\gamma_1}(1 - e^{-\gamma_1 t}), \quad (13)$$

$$\frac{d}{dt} y_1(t) + \gamma_1 y_1(t) + \mu_0 \|D^{m+k}v\|^2 \leq C_1, \quad (11)$$

$$\mu_0 \int_0^T \|D^{m+k} v\|^2 dt \leq y_1(0) + \int_0^T C_1 dt, \quad (14)$$

So, we have

$$\|(u, v)\|_{E_k}^2 = \|D^{2m+k} u\|^2 + \|D^k v\|^2 \leq y_1(0)e^{-\gamma_1 t} + \frac{C_1}{\gamma_1} (1 - e^{-\gamma_1 t}), \quad (15)$$

and

$$\lim_{t \rightarrow \infty} \|(u, v)\|_{E_k}^2 \leq \frac{C_1}{\gamma_1} = R_k^2 \quad (t > t_k). \quad (16)$$

Thus, there exist a non-negative constant R_k^2 and $t_k = t_k(\Omega) > 0$, such that

$$\|(u, v)\|_{E_k}^2 = \|D^{2m+k} u\|^2 + \|D^k v\|^2 \leq R_k^2 \quad (t > t_k). \quad (17)$$

Lemma 1 is proved

Theorem 1 Under the hypotheses of Lemma 1, and $I(x) \in H$, $(u_0, u_1) \in E_k$. So the initial boundary value

$$(u_{st}(t) + a(t)(-\Delta)^m u_{st}(t) + \Delta^{2m} u_s(t) + N(\|D^m u_s(t)\|_q^q)(-\Delta)^m u_s(t), w_j) = (I(x), w_j) \quad (1 \leq j \leq s). \quad (18)$$

Where, (18) meet the initial conditions $u_s(0) = u_{0s}$, $u_{st}(0) = u_{1s}$.

When $s \rightarrow +\infty$, we can obtain $(u_{0s}, u_{1s}) \rightarrow (u_0, u_1)$ in E_k . According to the basic theory of solutions of ordinary differential equations, we know that the approximate solution $u_s(t)$ exists on $(0, t_s)$. Step 2: prior estimation

In order to prove the existence of weak solution in space E_k ($k = 0, 1, 2, 3, \dots, 2m$), we multiply $g'_{js}(t) + \varepsilon g_{js}(t)$ at bothsides of equation (18) and sum of j . Set $v_s(t) = u_{st}(t) + \varepsilon u_s(t)$.

When $k = 0, 1, 2, 3, \dots, 2m$, we get a priori estimate of solution in space E_k

$$\|(u_s, v_s)\|_{E_k}^2 = \|D^{2m+k} u_s\|^2 + \|D^k v_s\|^2 \leq R_k^2, \quad (19)$$

$$\mu_0 \int_0^T \|D^{m+k} v\|^2 dt \leq y_1(0) + \int_0^T C_1 dt. \quad (20)$$

problem (1) - (3) has a unique global solution $(u, v) \in L^\infty([0, +\infty); E_k)$.

Proof. Existence: the existence of global solution is proved by Galerkin method

Step 1: construct approximate solution

Let $(-\Delta)^{m+z} w_j = \lambda_j^{m+z} w_j$, $z = 0, 1, 2, 3, \dots, m$, where λ_j is the eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary on Ω , w_j is the eigenfunction determined by the corresponding eigenvalue λ_j . all the eigenfunctions $\{w_j\}_{j=1}^\infty$ constitute the standard orthogonal basis of $H_0^{2m}(\Omega) \cap H(\Omega)$.

Set $u_s = u_s(t) = \sum_{j=1}^s g_{js}(t) w_j$ be the approximate solution

of the initial boundary value problem (1) - (3), where $g_{js}(t)$ is determined by the following system of differential equations

It can be seen that the priori estimates of the solution of lemma 1 in formula (19) and (20) hold respectively. It is known that (u_s, v_s) is bounded in $L^\infty([0, +\infty); E_k)$ by formula (19) and can be obtained $v_s \in L^2(0, T; H_0^{m+k}(\Omega))$ by formula (20).

Step 3: limit process

In space E_k ($k = 0, 1, 2, 3, \dots, 2m$), the subsequence $\{u_l\}$ is selected from sequence $\{u_s\}$, so that $(u_l, v_l) \rightarrow (u, v)$ converges weakly * in $L^\infty([0, +\infty); E_k)$. In addition, from formula (20), we can get that u_l is bounded in $L^2(0, T; H_0^{m+k}(\Omega))$.

If we know that E_k is closely embedded in E_0 by rellich-kondrachov compact embedding theorem, then $(u_l, v_l) \rightarrow (u, v)$ converges strongly almost everywhere in E_0 . Thus, we can make $s = l$ and take the limit in equation (18). For fixed j , and $l \geq j$, we can get from formula (18).

$$(u_{lt}(t) + a(t)(-\Delta)^m u_{lt}(t) + \Delta^{2m} u_l(t) + N(\|D^m u_l(t)\|_q^q)(-\Delta)^m u_l(t), w_j) = (I(x), w_j). \quad (21)$$

Because $u_s \rightarrow u$ weakly * converges in $L^\infty([0, +\infty); H^{2m+k}(\Omega) \cap H_0^1(\Omega))$, then

B weakly* converges in C

$$(u_l(t), (-\Delta)^k w_j) \rightarrow (u(t), \lambda_j^k w_j) \text{ weakly* converges in } L^\infty[0, +\infty),$$

$$(u_{lt}(t), (-\Delta)^k w_j) \rightarrow (u_l(t), \lambda_j^k w_j) \text{ weakly* converges in } L^\infty[0, +\infty).$$

Therefore, $(u_{tt}(t), (-\Delta)^k w_j) = \frac{d}{dt}(u_{tt}(t), (-\Delta)^k w_j) \rightarrow (u_{tt}(t), \lambda_j^k w_j)$ converges in $D'[0, +\infty)$, $D'[0, +\infty)$ is the conjugate space of $D[0, +\infty)$ infinitely differentiable space

$$(\Delta^{2m} u_l(t), (-\Delta)^k w_j) \rightarrow (\Delta^{2m} u(t), \lambda_j^k w_j) \text{ weakly* converges in } L^\infty[0, +\infty).$$

As a result, $((-\Delta)^m u_{tt}(t), (-\Delta)^k w_j) = ((-\Delta)^{\frac{k}{2}} v_l(t), (-\Delta)^{\frac{2m+k}{2}} w_j) - \varepsilon ((-\Delta)^{\frac{m+k}{2}} u_l(t), (-\Delta)^{\frac{m+k}{2}} w_j)$, then $a(t)((-\Delta)^m u_{tt}(t), (-\Delta)^k w_j) \rightarrow a(t)[((-\Delta)^{\frac{k}{2}} v(t), \lambda_j^{\frac{2m+k}{2}} w_j) - \varepsilon ((-\Delta)^{\frac{m+k}{2}} u(t), \lambda_j^{\frac{m+k}{2}} w_j)]$ weakly* converges in $L^\infty[0, +\infty)$.

Further, $(N(\|D^m u_l(t)\|_q^q)(-\Delta)^m u_l(t), (-\Delta)^k w_j) = (N(\|D^m u_l(t)\|_q^q)(-\Delta)^{\frac{m+k}{2}} u_l(t), \lambda_j^{\frac{m+k}{2}} w_j)$, then $(N(\|D^m u_l(t)\|_q^q)(-\Delta)^m u_l(t), (-\Delta)^k w_j) \rightarrow (N(\|D^m u(t)\|_q^q)(-\Delta)^{\frac{m+k}{2}} u(t), \lambda_j^{\frac{m+k}{2}} w_j)$ weakly* converges in $L^\infty[0, +\infty)$.

In particular, $u_{0l} \rightarrow u_0$ weakly converges in E_k ; $u_{lt} \rightarrow u_l$ weakly converges in E_k .

For any j and $l \rightarrow +\infty$, we can get

$$(u_{tt}(t) + a(t)(-\Delta)^m u_t(t) + \Delta^{2m} u(t) + N(\|D^m u(t)\|_q^q)(-\Delta)^m u(t), (-\Delta)^k w_j) = (I(x), (-\Delta)^k w_j) \quad (1 \leq j \leq s).$$

Because of the arbitrariness of w_j , we have for any $B \ v \in H^k(\Omega) \cap H_0^1(\Omega)$

$$(u_{tt}(t) + a(t)(-\Delta)^m u_t(t) + \Delta^{2m} u(t) + N(\|D^m u(t)\|_q^q)(-\Delta)^m u(t), v) = (I(x), v),$$

Therefore, the existence is proved.

Next, we prove the uniqueness of the solution.

Let u^*, v^* be two solutions of the system of equations, let $w = u^* - v^*$, then w satisfies

$$w_{tt} + a(t)(-\Delta)^m w_t + \Delta^{2m} w + N(\|D^m u^*(t)\|_q^q)(-\Delta)^m u^* - N(\|D^m v^*(t)\|_q^q)(-\Delta)^m v^* = 0, \quad (22)$$

$$w(0) = 0, \quad w'(0) = 0, \quad x \in \Omega \subset R^n. \quad (23)$$

By using the inner product in H of equation (1) with $w_t + \varepsilon w$, we obtain

$$(w_{tt} + a(t)(-\Delta)^m w_t + \Delta^{2m} w + N(\|D^m u^*(t)\|_q^q)(-\Delta)^m u^* - N(\|D^m v^*(t)\|_q^q)(-\Delta)^m v^*, w_t + \varepsilon w) = 0, \quad (24)$$

By using Hölder inequality, Young inequality and Poincaré inequality, the following are obtained by dealing with the following items in formula (22)

$$(w_{tt}, w_t + \varepsilon w) = \frac{1}{2} \frac{d}{dt} \|w_t\|^2 + \varepsilon \frac{d}{dt} (w_t, w) - \varepsilon \|w_t\|^2. \quad (25)$$

$$(\Delta^{2m} w, w_t + \varepsilon w) = \frac{1}{2} \frac{d}{dt} \|(-\Delta)^m w\|^2 + \varepsilon \|(-\Delta)^m w\|^2. \quad (26)$$

According to the hypothesis (A1) - (A2), Poincaré inequality and differential mean value theorem, we can obtain

$$(a(t)(-\Delta)^m w_t, w_t + \varepsilon w) \geq \frac{\varepsilon \mu}{2} \frac{d}{dt} \|D^m w\|^2 + \mu_0 \|D^m w_t\|^2. \quad (27)$$

$$(N(\|D^m u^*(t)\|_q^q)(-\Delta)^m u^* - N(\|D^m v^*(t)\|_q^q)(-\Delta)^m v^*, w_t + \varepsilon w)$$

$$= (N(\|D^m u^*(t)\|_q^q)(-\Delta)^m u^* - N(\|D^m v^*(t)\|_q^q)(-\Delta)^m v^*, w_t)$$

$$+ (N(\|D^m u^*(t)\|_q^q)(-\Delta)^m u^* - N(\|D^m v^*(t)\|_q^q)(-\Delta)^m v^*, \varepsilon w)$$

$$= I_1 + \varepsilon I_2. \quad (28)$$

By using Sobolev embedding theorem, we can obtain that, $H_0^m(\Omega) \subset L^q(\Omega)$, then there is a constant $K > 0$, such that

$$\|D^m u^*(t)\|_{L^q(\Omega)}^q \leq K \|(-\Delta)^m u^*(t)\|^{\frac{(2m+n)q-2n}{4m}} \|u\|^{\frac{2mq-nq+2n}{4m}}, \quad (29)$$

where

$$\begin{cases} \frac{2n}{n+2m} \leq q \leq \frac{2n}{n-2m}, & n > 2m; \\ \frac{2n}{n+2m} \leq q < \infty, & n \leq 2m. \end{cases} \quad I_2 \geq \delta_1 \|D^m w\|^2 - \frac{C_4}{2} \|(-\Delta)^m w\|^2 - \frac{C_3^2}{2} \|D^m w\|^2. \quad (31)$$

Combined with formula (24) - (31), it is concluded that

$$\begin{aligned} \text{Therefore,} \quad \frac{d}{dt} (\|w_t\|^2 + \|(-\Delta)^m w\|^2 + (\varepsilon\mu + \delta) \|D^m w\|^2 + 2\varepsilon(w_t, w)) &\leq 2\varepsilon \|w_t\|^2 \\ I_1 = (N(\|D^m u^*(t)\|_q^q) (-\Delta)^m u^* - N(\|D^m v^*(t)\|_q^q) (-\Delta)^m v^*, w_t) &+ (\frac{C_2}{2\mu_0} + \varepsilon C_4 - 2\varepsilon) \|(-\Delta)^m w\|^2 + (\varepsilon C_3^2 - 2\varepsilon\delta_1) \|D^m w\|^2. \quad (32) \\ \geq \frac{\delta}{2} \frac{d}{dt} \|D^m w\|^2 - |N'(\xi)| (\sum_{i=0}^{q-1} \|D^m u^*(t)\|_q^i \|D^m v^*(t)\|_q^{q-i-1}) \|D^m w\| \|D^m v^*\| \|D^m w_t\| &\text{Further, due to} \\ \geq \frac{\delta}{2} \frac{d}{dt} \|D^m w\|^2 - \frac{C_2}{4\mu_0} \|(-\Delta)^m w\|^2 - \mu_0 \|D^m w_t\|^2. \quad (30) & (w_t, w) \geq -\frac{1}{2} \|w_t\|^2 - \frac{\lambda_1^{-m}}{2} \|D^m w\|^2, \quad (33) \end{aligned}$$

so,

Similarly, there are

$$\frac{d}{dt} (\|w_t\|^2 + \|(-\Delta)^m w\|^2 + (\varepsilon\mu + \delta) \|D^m w\|^2 + 2\varepsilon(w_t, w)) \leq 3\varepsilon \|w_t\|^2 + (\frac{C_2}{2\mu_0} + \varepsilon C_4 - 2\varepsilon) \|(-\Delta)^m w\|^2 + \varepsilon(C_3^2 - 2\delta_1 + \lambda_1^{-m}) \|D^m w\|^2 + 2\varepsilon(w_t, w). \quad (34)$$

$$\text{Let } \gamma_2 = \max \left\{ 3\varepsilon, \frac{C_2}{2\mu_0} + \varepsilon C_4 - 2\varepsilon, \frac{\varepsilon(C_3^2 - 2\delta_1 + \lambda_1^{-m})}{(\varepsilon\mu + \delta)}, 1 \right\}, \text{ such that}$$

$$\frac{d}{dt} y_2(t) \leq \gamma_2 y_2(t), \quad (35)$$

where

$$y_2(t) = \|w_t\|^2 + \|(-\Delta)^m w\|^2 + (\varepsilon\mu + \delta) \|D^m w\|^2 + 2\varepsilon(w_t, w), \quad (36)$$

By Gronwall inequality, we get

$$y_2(t) \leq y_2(0)e^{\gamma_2 t} = 0, \quad (37)$$

Thus $y_2(t) = 0$, i.e. $u^* = v^*$, so the uniqueness is proved.

Theorem 2 According to lemma 1 and theorem 1, then the initial boundary value problem (1) - (3) has a family of global attractors

$$A_k = \omega(B_{0k}) = \overline{\bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} S(t_k) B_{0k}}, \quad (k = 1, 2, 3, \dots, 2m),$$

where $B_{0k} = \{(u, v) \in E_k : \|(u, v)\|_{E_k}^2 = \|D^{2m+k} u\|^2 + \|D^k v\|^2 \leq R_k^2 + R_0^2\}$ is a bounded absorbing set in E_k and satisfies the following conditions:

- (i) $S(t)A_k = A_k, t > 0$;
- (ii) $\lim_{t \rightarrow \infty} \text{dist}(S(t)B_k, A_k) = 0 \quad (\forall B_k \subset E_k)$ B_k is a bounded set;

where $\text{dist}(S(t)B_k, A_k) = \sup_{x \in B_k} \inf_{y \in A_k} \|S(t)x - y\|_{E_k}$, $S(t)$ is the solution semigroup generated by the initial boundary value problem (1) - (3).

Proof. It is necessary to verify the conditions (I), (II) and (III) for the existence of attractors in reference [2]. Under the condition of Theorem 1, there exists a solution semigroup $S(t): E_k \rightarrow E_k$ of the initial boundary value problem (1) - (3).

From lemma 1, we can obtain that $\forall B_k \subset E_k$ is a bounded set that includes in the ball $\{(u, v) \in E_k : \|(u, v)\|_{E_k} \leq R_k\}$.

$$\|S(t)(u_0, v_0)\|_{E_k}^2 = \|u\|_{H_0^{2m+k}(\Omega)}^2 + \|v\|_{H_0^k(\Omega)}^2 \leq \|u_0\|_{H_0^{2m+k}(\Omega)}^2 + \|v_0\|_{H_0^k(\Omega)}^2 + C \leq R_k^2 + C, \quad (38)$$

where $t \geq 0$, $(u_0, v_0) \in B_k$, this shows that $\{S(t)\} (t \geq 0)$ is uniformly bounded in E_k .

Furthermore, for any $(u_0, v_0) \in E_k$, when $t \geq \max\{t_1, t_k\}$, we have

$$\|S(t)(u_0, v_0)\|_{E_k}^2 = \|u\|_{H_0^{2m+k}(\Omega)}^2 + \|v\|_{H_0^k(\Omega)}^2 \leq R_k^2 + R_0^2, \quad (39)$$

Therefore,

$$B_{0k} = \left\{ (u, v) \in E_k : \| (u, v) \|_{E_k}^2 = \| D^{2m+k} u \|^2 + \| D^k v \|^2 \leq R_k^2 + R_0^2 \right\}$$

is a bounded absorbing set in semigroup $S(t)$.

According to the Rellich Kondrachov compact embedding theorem, if E_k is compactly embedded in E_0 , then the bounded set in E_k is the compact set in E_0 . Therefore, the solution semigroup $S(t)$ is a completely continuous operator, thus the global attractor family A_k of solution semigroup $S(t)$ is obtained. Where $A_k = \omega(B_{0k}) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t)B_{0k}}$.

The prove is completed.

4. Dimension Estimation

In this part, we first linearize the equation into a first-order variational equation and prove that the solution semigroup $s(t)$ is Fréchet differentiable on E_k . Furthermore, we prove the decay of the volume element of the linearization problem. Finally, we estimate the upper bound of the Hausdorff dimension and fractal dimension of A_k .

The initial boundary value problem (1) - (3) is linearized and rewritten into a first order variational equation

$$w_t + \varepsilon w = \theta, \quad (40)$$

$$\theta_t - \varepsilon \theta + \varepsilon^2 w + \Delta^{2m} w + a(t)(-\Delta)^m \theta - \varepsilon a(t)(-\Delta)^m w + N(\|D^m u(t)\|_q^q)(-\Delta)^m w$$

$$+ N'(\|D^m u(t)\|_q^q)(\|D^m u(t)\|_q^q)' D^m w (-\Delta)^m u = 0, \quad (41)$$

$$h = [N(\|D^m \bar{u}(t)\|_q^q) - N(\|D^m u(t)\|_q^q)](-\Delta)^m \bar{u} - N'(\|D^m u(t)\|_q^q)(\|D^m u(t)\|_q^q)' D^m w (-\Delta)^m u. \quad (49)$$

By using the inner product in E_k of Equation (47) with ϕ , we obtain

$$\begin{aligned} \frac{d}{dt} (\|D^k \phi\|^2 + \varepsilon^2 \|D^k \psi\|^2 + \|D^{2m+k} \psi\|^2) + 2\varepsilon^3 \|D^k \psi\|^2 + 2\varepsilon \|D^{2m+k} \psi\|^2 &= 2\varepsilon \|D^k \phi\|^2 \\ + 2N(\|D^m u(t)\|_q^q)(D^{2m+k} \psi, D^k \phi) + 2(-h, (-\Delta)^k \phi). \end{aligned} \quad (50)$$

Now we deal with h , let $f = D^m u(t)$, $\bar{f} = D^m \bar{u}(t)$, and deal with equation (49) with the help of differential mean value

$$(w, \theta)|_{t=0} = \bar{\xi}, \quad x \in \Omega, \quad (42)$$

$$w(x, t)|_{\partial\Omega} = (-\Delta)^i w(x, t)|_{\partial\Omega} = 0, \quad (43)$$

$$\theta(x, t)|_{\partial\Omega} = (-\Delta)^i \theta(x, t)|_{\partial\Omega} = 0, \quad i = 1, 2, \dots, 2m-1, \quad t > 0. \quad (44)$$

Where $\bar{\xi} = (\xi, \eta) \in E_k$, $(u, v) = S(t)(u_0, v_0)$ is the solution of the initial boundary value problem (1) - (3).

Given $(u_0, v_0) \in A_k$, then we can get $S(t)(u_0, v_0) \in E_k$. It is proved that there exists a unique solution to the linearized initial boundary value problem (40) - (44) for any $(\xi, \eta) \in E_k$.

Lemma 2 If $S(t): E_k \rightarrow E_k$, the Frechet differential on $\eta_0 = (u_0, v_0)$ is a linear operator $F: (\xi, \eta) \rightarrow (w, \theta)$, for any $t > 0, R > 0$, the mapping $S(t): E_k \rightarrow E_k$ is Fréchet differentiable on E_k , where (w, θ) is the solution of the linearized initial boundary value problem (40) - (44).

Proof. Set $\eta_0 = (u_0, v_0) \in E_k$, $\bar{\eta}_0 = (u_0 + \xi, v_0 + \eta) \in E_k$ and $\|\eta_0\|_{E_k} \leq R$, $\|\bar{\eta}_0\|_{E_k} \leq R$.

We define $\eta_1 = S(t)\eta_0 = (u, v)$, $\bar{\eta}_1 = S(t)\bar{\eta}_0 = (\bar{u}, \bar{v})$, where the semigroup $S(t)$ is Lipschitz continuous on the bounded set of E_k , i.e.

$$\|S(t)\eta_0 - S(t)\bar{\eta}_0\|_{E_k}^2 \leq e^{ct} \|(\xi, \eta)\|_{E_k}^2. \quad (45)$$

Let $(\psi, \phi) = \bar{\eta}_1 - \eta_1 - U = (\bar{u} - u - w, \bar{v} - v - \theta)$, then

$$\psi_t + \varepsilon \psi = \phi, \quad (46)$$

$$\phi_t - \varepsilon \phi + \varepsilon^2 \psi + \Delta^{2m} \psi + N(\|D^m u(t)\|_q^q)(-\Delta)^m \psi = -h, \quad (47)$$

$$\psi(0) = \phi(0) = 0, \quad (48)$$

where

theorem

$$\begin{aligned} h &= N'(\|\varsigma_1\|_q^q)(\|\varsigma_1\|_q^q)' D^m(\bar{u}-u)(-\Delta)^m \bar{u} - N'(\|f\|_q^q)(\|f\|_q^q)' D^m(\bar{u}-u)(-\Delta)^m u \\ &\quad + N'(\|f\|_q^q)(\|f\|_q^q)'(-\Delta)^m u D^m \psi \\ &= h_1 + h_2, \end{aligned} \quad (51)$$

Where $\varsigma_1 = (1-s)\bar{f} + sf$, $s \in (0,1)$, $h_1 = N'(\|f\|_q^q)(\|f\|_q^q)'(-\Delta)^m u D^m \psi$,

$$h_2 = N'(\|\varsigma_1\|_q^q)(\|\varsigma_1\|_q^q)' D^m(\bar{u}-u)(-\Delta)^m \bar{u} - N'(\|f\|_q^q)(\|f\|_q^q)' D^m(\bar{u}-u)(-\Delta)^m u.$$

Further, let $g'(\varsigma_1) = N'(\|\varsigma_1\|_q^q)(\|\varsigma_1\|_q^q)'$, then

$$\begin{aligned} h_2 &= [(g'(\varsigma_1) - g'(f))(-\Delta)^m \bar{u} + g'(f)(-\Delta)^m(\bar{u}-u)] D^m(\bar{u}-u) \\ &= g''(\varsigma_2)(1-s)(D^m(\bar{u}-u))^2(-\Delta)^m \bar{u} + g'(f)(-\Delta)^m(\bar{u}-u) D^m(\bar{u}-u), \end{aligned} \quad (52)$$

where $\varsigma_2 = \theta\varsigma_1 + (1-\theta)f$, $\theta \in (0,1)$.

Therefore,

$$|(-h, (-\Delta)^k \phi)| = |(-(h_1 + h_2), (-\Delta)^k \phi)| \leq |(h_1, (-\Delta)^k \phi)| + |(h_2, (-\Delta)^k \phi)|, \quad (53)$$

where

$$\begin{aligned} |(h_1, (-\Delta)^k \phi)| &= |(N'(\|f\|_q^q)(\|f\|_q^q)'(-\Delta)^m u D^m \psi, (-\Delta)^k \phi)| \\ &\leq C_5 \|(-\Delta)^m u\|_\infty \|D^{m+k} \psi\| \|D^k \phi\| \\ &\leq \frac{C_6 \lambda_1^m}{2} \|D^{2m+k} \psi\|^2 + \frac{C_6}{2} \|D^k \phi\|^2. \end{aligned} \quad (54)$$

$$\begin{aligned} |(h_2, (-\Delta)^k \phi)| &= |(g''(\varsigma_2)(1-s)(D^m(\bar{u}-u))^2(-\Delta)^m \bar{u} + g'(f)(-\Delta)^m(\bar{u}-u) D^m(\bar{u}-u), (-\Delta)^k \phi)| \\ &\leq \|g''(\varsigma_2)\|_\infty \left| \int_\Omega (-\Delta)^m \bar{u} (D^m(\bar{u}-u))^2 (-\Delta)^k \phi dx \right| + \|g'(f)\|_\infty \left| \int_\Omega (-\Delta)^m (\bar{u}-u) D^m(\bar{u}-u) (-\Delta)^k \phi dx \right| \\ &\leq C_7 \|D^k \phi\| \left(\int_\Omega [D^{2m+k} \bar{u} (D^m(\bar{u}-u))^2 + D^{2m} \bar{u} D^k (D^m(\bar{u}-u))^2] dx \right)^{\frac{1}{2}} \\ &\quad + C_8 \|D^k \phi\| \left(\int_\Omega [D^{2m+k} (\bar{u}-u) D^m(\bar{u}-u) + D^{2m} (\bar{u}-u) D^k (D^m(\bar{u}-u))]^2 dx \right)^{\frac{1}{2}} \\ &\leq 4C_7 \|D^k \phi\| \left(\int_\Omega (D^{2m+k} \bar{u})^2 (D^m(\bar{u}-u))^4 dx + \int_\Omega (D^{2m} \bar{u})^2 (D^{m+k}(\bar{u}-u))^4 dx \right)^{\frac{1}{2}} \\ &\quad + 4C_8 \|D^k \phi\| \left(\int_\Omega (D^{2m+k} (\bar{u}-u))^2 (D^m(\bar{u}-u))^2 dx + \int_\Omega (D^{2m} (\bar{u}-u))^2 (D^{m+k}(\bar{u}-u))^2 dx \right)^{\frac{1}{2}} \\ &\leq 4C_7 \|D^k \phi\| (\|D^m(\bar{u}-u)\|_\infty^2 \|D^{2m+k} \bar{u}\| + \|D^{m+k}(\bar{u}-u)\|_\infty^2 \|D^{2m} \bar{u}\|) \\ &\quad + 4C_8 \|D^k \phi\| (\|D^{2m+k}(\bar{u}-u)\|_\infty \|D^m(\bar{u}-u)\| + \|D^{2m}(\bar{u}-u)\|_\infty \|D^{m+k}(\bar{u}-u)\|) \end{aligned}$$

$$\begin{aligned} &\leq 4C_9 \|D^k \phi\| \|D^{2m+k}(\bar{u}-u)\|^2 + 4C_{10} \|D^k \phi\| \|D^{2m+k}(\bar{u}-u)\|^2 \\ &\leq \frac{C_9 + C_{10}}{2} \|D^k \phi\|^2 + 2(C_{11}^2 + C_{12}^2) \|D^{2m+k}(\bar{u}-u)\|^4. \end{aligned} \quad (55)$$

Combined with formula (53) - (55), it is concluded that

$$|(-h, (-\Delta)^k \phi)| \leq \frac{C_6 \lambda_1^m}{2} \|D^{2m+k} \psi\|^2 + \frac{C_6 + C_9 + C_{10}}{2} \|D^k \phi\|^2 + 2(C_{11}^2 + C_{12}^2) \|D^{2m+k}(\bar{u}-u)\|^4. \quad (56)$$

Furthermore, substituting formula (56) into equation (50), the result is as follows

$$\begin{aligned} \frac{d}{dt} (\|D^k \phi\|^2 + \varepsilon^2 \|D^k \psi\|^2 + \|D^{2m+k} \psi\|^2) &\leq (\delta_1 + C_6 + C_9 + C_{10} + 2\varepsilon) \|D^k \phi\|^2 - 2\varepsilon^3 \|D^k \psi\|^2 \\ &\quad + (\delta_1 + C_6 \lambda_1^m - 2\varepsilon) \|D^{2m+k} \psi\|^2 + 4(C_{11}^2 + C_{12}^2) \|D^{2m+k}(\bar{u}-u)\|^4. \end{aligned} \quad (57)$$

Let $\gamma_3 = \max\{\delta_1 + C_6 + C_9 + C_{10} + 2\varepsilon, -2\varepsilon, \delta_1 + C_6 \lambda_1^m - 2\varepsilon, 4(C_{11}^2 + C_{12}^2)\}$, we can get

$$\frac{d}{dt} (\|D^k \phi\|^2 + \varepsilon^2 \|D^k \psi\|^2 + \|D^{2m+k} \psi\|^2) \leq \gamma_3 (\|D^k \phi\|^2 + \varepsilon^2 \|D^k \psi\|^2 + \|D^{2m+k} \psi\|^2 + \|D^{2m+k}(\bar{u}-u)\|^4).$$

From Gronwall inequality and Lipschitz property of (45), we can obtain

$$\|D^k \phi\|^2 + \|D^{2m+k} \psi\|^2 \leq e^{(\gamma_3 + C_{13})t} \|(\xi, \eta)\|_{E_k}^4, \quad (58)$$

So, when $\|(\xi, \eta)\|_{E_k} \rightarrow 0$, Fréchet differentiable, then the linearized first order variational equation (41) can be rewritten as

$$\frac{\|S(t)\eta_0 - S(t)\bar{\eta}_0 - FS(t)(\xi, \eta)\|_{E_k}^2}{\|(\xi, \eta)\|_{E_k}^2} \leq e^{(\gamma_3 + C_{13})t} \|(\xi, \eta)\|_{E_k}^2 \rightarrow 0. \quad (59)$$

$$P_t + \Lambda(\vartheta)P = 0, \quad (59)$$

$$P_t = F_t(\vartheta), \quad (60)$$

The prove is completed.

Theorem 3 Under the assumptions and conditions of theorem 2, then a family of global attractors A_k of initial boundary value problem (1) - (3) has Hausdorff dimension and fractal dimension, and

$$d_H(A_k) < \frac{1}{5}n, \quad d_F(A_k) < \frac{6}{5}n.$$

Proof. Let $\vartheta = R_\varepsilon \varphi = (u, v)^T$, $\varphi = (u, u_t)^T$, $v = u_t + \varepsilon u$, then $R_\varepsilon : \{u, u_t\} \rightarrow \{u, u_t + \varepsilon u\}$ is an isomorphic mapping. If $A_i (i = 1, 2, \dots, 2m)$ is the global attractor of $\{S(t)\}$, then $A_{\varepsilon i}$ is the global attractor of $\{S_\varepsilon(t)\}$, and they have the same dimension.

From lemma 2, we can get that $S(t) : E_k \rightarrow E_k$ is

where

$$\Lambda(\vartheta) = \begin{pmatrix} \varepsilon I & -I \\ (N(\|A^{\frac{m}{2}} u\|_q^q) - \varepsilon a(t))A^m + A^{2m} + \varepsilon^2 I + \Phi & a(t)A^m - \varepsilon I \end{pmatrix},$$

I is an identity operator, $-\Delta = A$, $P = (w, \theta)^T \in E_k$, $\theta = w_t + \varepsilon w$.

For a fixed $(u_0, v_0) \in E_k$, let $\beta_1, \beta_2, \dots, \beta_n$ be n elements of E_k , and $w_1(t), w_2(t), \dots, w_n(t)$ be n solutions of linear equation (59), whose initial value is $w_1(0) = \beta_1, w_2(0) = \beta_2, \dots, w_n(0) = \beta_n$.

Therefore,

$$\frac{d}{dt} \|w_1(t) \Lambda w_2(t) \Lambda \dots \Lambda w_n(t)\|_{\Lambda E_k}^2 - 2 \text{tr} F_t(\vartheta(\tau)) \cdot Q_n(\tau) \|\beta_1 \Lambda \beta_2 \Lambda \dots \Lambda \beta_n\|_{\Lambda E_k} = 0. \quad (61)$$

Furthermore, from the uniform Gronwall inequality, we can obtain

$$\|w_1(t) \Lambda w_2(t) \Lambda \dots \Lambda w_n(t)\|_{\Lambda E_k}^2 = \|\beta_1 \Lambda \beta_2 \Lambda \dots \Lambda \beta_n\|_{\Lambda E_k} \exp\left(\int_0^t \text{tr} F_t(\vartheta(\tau)) \cdot Q_n(\tau) d\tau\right), \quad (62)$$

where Λ is the outer product, tr is the trace, $Q_n(\tau)$ is an orthogonal projection from space E_k to $span\{w_1(t), w_2(t), \dots, w_n(t)\}$.

For a given time τ , let $\omega_j(\tau) = (\xi_j(\tau), \eta_j(\tau))^T$ ($j=1, 2, \dots, n$) be the standard orthogonal basis of space $span\{w_1(t), w_2(t), \dots, w_n(t)\}$.

We define the inner product of E_k as

$$((\xi, \eta), (\bar{\xi}, \bar{\eta})) = ((D^{2m+k}\xi, D^{2m+k}\bar{\xi}) + (D^k\eta, D^k\bar{\eta})). \quad (63)$$

To sum up, it can be concluded that

$$tr F_t(\vartheta(\tau)) \cdot Q_n(\tau) = \sum_{j=1}^N (F_t(\vartheta(\tau)) \cdot Q_n(\tau) \omega_j(\tau), \omega_j(\tau))_{E_k} = \sum_{j=1}^N (F_t(\vartheta(\tau)) \omega_j(\tau), \omega_j(\tau))_{E_k}, \quad (64)$$

where

$$\begin{aligned} & (F_t(\vartheta(\tau)) \omega_j(\tau), \omega_j(\tau))_{E_k} = -(\Lambda(\vartheta) \omega_j, \omega_j) \\ & \leq -\varepsilon \|D^{2m+k}\xi_j\|^2 + \frac{1}{2}(\delta_1 - \varepsilon\mu_0)(\|D^{2m+k}\xi_j\|^2 + \|D^k\eta_j\|^2) + \frac{\varepsilon^2\lambda_1^{-2m}}{2} \|D^{2m+k}\xi_j\|^2 \\ & + \frac{\varepsilon^2}{2} \|D^k\eta_j\|^2 - \lambda_1^m\mu_0 \|D^k\eta_j\|^2 + \varepsilon \|D^k\eta_j\|^2 + \frac{\varepsilon}{2} \|D^{2m+k}\xi_j\|^2 + \frac{C_{14}\lambda_1^{-m-k}}{2\varepsilon} \|D^k\eta_j\|^2 \\ & \leq \frac{(\delta_1 - \varepsilon\mu_0) + \varepsilon^2\lambda_1^{-2m}}{2} \|D^{2m+k}\xi_j\|^2 + (\frac{\varepsilon^2}{2} + \frac{C_{14}\lambda_1^{-m-k}}{2\varepsilon}) \|D^k\eta_j\|^2 + r \|D^k\eta_j\|^2 \\ & \leq -\frac{C_{15}}{2} (\|D^{2m+k}\xi_j\|^2 + \|\nabla^k\eta_j\|^2) + r \|D^k\eta_j\|^2, \end{aligned} \quad (65)$$

where $C_{15} = \min\left\{(\varepsilon\mu_0 - \delta_1) - \varepsilon^2\lambda_1^{-2m}, -\varepsilon^2 - \frac{C_{14}\lambda_1^{-m-k}}{\varepsilon}\right\}$,

$$r = \max\{2\varepsilon, \delta_1\}.$$

Owing to the $\omega_j(\tau) = (\xi_j(\tau), \eta_j(\tau))^T$, $j=1, 2, \dots, n$ is the standard orthonormal basis of $span\{w_1(t), w_2(t), \dots, w_n(t)\}$, so

$$\|D^{2m+k}\xi_j\|^2 + \|D^k\eta_j\|^2 = 1, \quad (66) \quad \text{and}$$

$$\sum_{j=1}^n (F_t(\vartheta(\tau)) \omega_j(\tau), \omega_j(\tau))_{E_k} \leq -\frac{nC_{15}}{2} + r \sum_{j=1}^n \|\nabla^k\eta_j\|^2, \quad (67)$$

Let

$$q_n(t) = \sup_{\vartheta_0 \in B_{0k}} \sup_{\substack{\eta_j \in E_k \\ \|D^k\eta_j\| \leq 1}} \frac{1}{t} F_t(S(\tau)\vartheta_0) \cdot Q_n(\tau) d\tau, \quad (70)$$

$$q_n = \lim_{t \rightarrow \infty} q_n(t), \quad (71)$$

so

For almost all t , there are

$$\sum_{j=1}^n \|\nabla^k\eta_j\|^2 \leq \sum_{j=1}^n \lambda_j^{a-1} \quad (68)$$

where $a = \frac{k}{2m}$ and $a \in [0, 1]$, λ_j is the eigenvalue of A^{2m}

and $\lambda_1 < \lambda_2 < \dots < \lambda_n$, thus

$$q_n \leq -\frac{nC_{15}}{2} + r \sum_{j=1}^n \lambda_j^{a-1}. \quad (72)$$

Therefore, the Lyapunov exponent $\kappa_1, \kappa_2, \dots, \kappa_n$ ($n > 1$) of B_{0k} is uniformly bounded, and

$$\kappa_1 + \kappa_2 + \dots + \kappa_n \leq -\frac{nC_{15}}{2} + r \sum_{j=1}^n \lambda_j^{a-1}, \quad (73)$$

such that

$$(q_j)_+ \leq -\frac{nC_{15}}{2} + r \sum_{j=1}^n \lambda_j^{a-1} \leq r \sum_{j=1}^n \lambda_j^{a-1} \leq \frac{nC_{15}}{12}, \quad (74)$$

$$q_n \leq -\frac{nC_{15}}{2} \left(1 - \frac{2r}{nC_{15}} \sum_{j=1}^n \lambda_j^{a-1} \right) \leq -\frac{5}{12} nC_{15}. \quad (75)$$

Further, we can get

$$\max_{1 \leq j \leq n} \frac{(q_j)_+}{|q_m|} \leq \frac{1}{5}. \quad (76)$$

Therefore, $d_H(A_k) < \frac{1}{5}n$, $d_F(A_k) < \frac{6}{5}n$, i.e. the Hausdorff dimension and Fractal dimension of a family of global attractors are finite.

5. Conclusions

On this paper, we studies the existence and uniqueness of the solution of the equation, when the order m and the order q of the Banach space $L^q(\Omega)$ meet certain conditions (Lemma 1 and Theorem 1). Then, we obtain the solution semigroup $S(t)$ generated by the equation has a family of global attractor A_k in space $E_k = (H^{2m+k}(\Omega) \cap H_0^1(\Omega)) \times H_0^k(\Omega)$ (Theorem 2). Finally, we get the Hausdorff dimension and Fractal dimension of a family of global attractors A_k are finite (Lemma 2 and Theorem 3).

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