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# Stability and Hopf Bifurcation Analysis of Delayed Rosenzweig-MacArthur Model with Prey Immigration

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**Abstract:** A delayed reaction Cdiffusion Rosenzweig-MacArthur model with a constant rate of prey immigration is considered. We derive the characteristic equation through partial differential equation theory, and by analyzing the distribution of the roots of the characteristic equation, the local stability of the positive equilibria is studied, and we get the conditions to determine the stability of the positive equilibria. Furthermore we find that Hopf bifurcation occurs near the positive equilibrium when the time delay passes some critical values, and we get the conditions under which the Hopf bifurcation occurs and so periodic solutions appear near the positive equilibria. By using the center manifold theory and normal form method, we derive an explicit algorithm for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions. Furthermore, some numerical simulations are carried out to illustrate the analytic results of our study.

**Keywords:** Delay, Stability, Bifurcation, Center Manifold, Normal Form

## 1. Introduction

Predator-prey systems (or consumer-resource systems) are basic differential equation models for describing the interactions between two species with a pair of positive-negative feedbacks. Furthermore, many ecological concepts such as diffusion, functional responses and time delays have been taken into consideration to gain more accurate description and better understanding [1-6]. A typical Rosenzweig-MacArthur model was put forward first in [7]. In this model the prey has a logistic growth and the predator has a Holling II functional response. In [8-12] the global stability are discussed. There are also many researches on the limit cycle of R-M model [13-16]. In [17], Malay Banerjeethe studied the existence of Turing patterns in this model and in the effect of the non-local interaction on the periodic travelling wave and spatio-temporal chaotic patterns. In [18], a fractional order Rosenzweig-MacArthur (R-M) model incorporating a prey refuge is constructed and analyzed in detail. Sugie et al discussed the existence and uniqueness of limit cycles in predator-prey systems with a constant immigration in [19].

In this paper, we study the reaction-diffusion system, Rosenzweig-MacArthur model with a constant immigration.

$$\frac{\partial u(t,x)}{\partial t} = D_1 \triangle u(t,x) + ru(t,x)\left[1-\frac{u(t,x)}{k}\right] - \frac{u(t,x)v(t,x)}{a+u(t,x)} + b\frac{\partial v(t,x)}{\partial t} = D_2 \triangle v(t,x) - dv(t,x) +$$

$$\frac{\mu u(t-\tau,x)v(t-\tau,x)}{a+u(t-\tau,x)}, \quad x\in\Omega,\ \ t>0 \frac{\partial u(t,x)}{\partial \nu} = \frac{\partial u(t,x)}{\partial \nu} = 0, \tag{1}$$

$$x\in\partial\Omega,\ \ t\geq 0 u(t,x)=\phi(t,x)\geq 0,\ \ v(t,x)=\psi(t,x)\geq 0,\ \ x\in\bar{\Omega},\ \ \theta\in[-\tau,0]$$

Where $\Omega = (0, \pi)$, the homogeneous Neumann boundary condition means the system is a closed one. $u(t,x), v(t,x)$ stand for the densities of the prey and predator at time $t > 0$ and a spatial position $x \in \Omega$, $D_1, D_2 > 0$ are the diffusion coefficients of the species. $a, b, d, k, r, \mu$ are all positive constants, and the meaning of them are the same as those in [19], and $\tau \geq 0$ is the constant delay due to the gestation of the predator.

It is easy to see that system (1) always has a boundary equilibrium

$$E_0(u_0, 0) = \left(\frac{k}{2} + \frac{\sqrt{k^2r^2 + 4bkr}}{2r}, 0\right)$$

When $\mu > d$ , we let

$$R_0 = \frac{bk(\mu - d)^2 + adkr(\mu - d)}{a^2d^2r}$$

then we can get that when $R_0 \leq 1$ system (1) has no positive equilibrium and $E_0(x_0, 0)$ is the unique equilibrium of (1) ; and when $R_0 > 1$ system (1.1) has a positive equilibrium $E_*(u_*, v_*)$ besides $E_0(u_0, 0)$,

$$E_*(u_*, v_*) = \left(\frac{ad}{\mu - d}, \frac{\mu}{d}\left[ru_*\left(1 - \frac{u_*}{k}\right) + b\right]\right)$$

By straightforward calculation, we get the characteristic equation of the linearizing system of (1) about $E_0$

$$\left[\lambda + D_1n^2 - r\left(1 - \frac{2u_0}{k}\right)\right]$$
$$\left[\lambda + D_2n^2 + d - \frac{\mu u_0}{a + u_0}e^{-\lambda\tau}\right] = 0$$

Similar to the discussion in [20], we know that $E_0$ is locally asymptotically stable when $R_0 < 1$, and unstable when $R_0 > 1$.

The rest of the present paper is organized as follows: In Section 2, we analyze the local stability of the positive equilibrium $E_*$, and the existence of Hopf bifurcation at $E_*$. In section 3, we determine the properties of the bifurcating periodic solution. In Section 4, some numerical simulations are carried out to illustrate the analytic results.

## 2. Local Stability of $E_*$ and the Existence of Hopf Bi-furcation

As shown in section 1, we know that system (1) has a positive equilibrium $E_*(u_*, v_*)$ when $R_0 > 1$. Move $E_*(u_*, v_*)$ to the origin $O(0, 0)$ , we can rewrite (1) as follow:

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= D_1\triangle u(t,x) + \alpha_1 u(t,x) - \alpha_2 v(t,x) + f_1(u,v)\frac{\partial v(t,x)}{\partial t} = D_2\triangle v(t,x) - \\ & dv(t,x) + \beta_1 u(t-\tau,x) + dv(t-\tau,x) + f_2(u,v)\frac{\partial u(t,x)}{\partial \nu} = \frac{\partial u(t,x)}{\partial \nu} = 0, \\ & x \in \partial\Omega,\ \ t \geq 0 u(t,x) = \phi(t,x) - u_*,\ \ v(t,x) = \psi(t,x) - v_*,\ \ x \in \bar{\Omega},\ \ \theta \in [-\tau, 0] \end{aligned} \tag{2}$$

where

$$\begin{aligned} \alpha_1 &= r - \frac{2ru_*}{k} - \frac{av_*}{(a+u_*)^2},\quad \alpha_2 = \frac{u_*}{a+u_*},\quad \beta_1 = \frac{a\mu v_*}{(a+u_*)^2} \\ f_1(u,v) &= -\frac{ru(t,x)^2}{k} + \frac{av_*u^2(t,x) - a(a+u_*)u(t,x)v(t,x)}{(a+u_*)^2(a+u_*+u(t,x))} \\ f_2(u,v) &= a\mu\frac{(a+u_*)u(t-\tau,x)v(t-\tau,x) - v_*u^2(t-\tau,x)}{(a+u_*)^2(a+u_*+u(t-\tau,x))} \end{aligned} \tag{3}$$

Denotinig $X = C([0,\pi], \mathbb{R}^2),\ C_\tau = C([-\tau, 0], X)$

$$u_1 = u(t,\cdot),\ \ u_2 = v(t,\cdot),\ \ U = (u_1, u_2)^T$$

then we can write (2) as an abstract ordinary differential equation in the phase space $C_\tau$

$$\dot{U}(t) = D\triangle U(t) + L(U_t) + F(U_t) \tag{4}$$

where

$$\begin{aligned} \mathrm{dom}(D\triangle) &= \{(u,v)^T \mid u, V \in C^2([0,\pi], \mathbb{R}),\ \partial_\nu u = \partial_\nu v = 0,\ \pi\} \\ L(\phi) &= \begin{pmatrix} \alpha_1 & -\alpha_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \beta_1 & d \end{pmatrix}\begin{pmatrix} \phi_1(-\tau) \\ \phi_2(-\tau) \end{pmatrix},\quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \\ F(\phi) &= \begin{pmatrix} -\dfrac{r\phi_1^2(0)}{k} + \dfrac{av_*\phi_1^2(0) - a(a+u_*)\phi_1(0)\phi_2(0)}{(a+u_*)^2(a+u_*+\phi_1(0))} \\ a\mu\dfrac{(a+u_*)\phi_1(-\tau)\phi_2(-\tau) - v_*\phi_1^2(-\tau)}{(a+u_*)^2(a+u_*+\phi_1(-\tau))} \end{pmatrix}. \end{aligned}$$

Equation (4) has the linear part

$$\dot{U}(t) = D\triangle U(t) + L(U_t)$$

and the characteristic equation

$$\lambda y - D\triangle y - L(\mathrm{e}^{\lambda \cdot} y) = 0, \quad y \in \mathrm{dom}(D\triangle)\backslash\{0\}$$

Letting

$$\phi = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos(n\pi x)$$

be the characteristic function corresponding to the eigenvalues $\lambda$ , then the characteristic equation has the following form

$$\triangle_n(\lambda, \tau) = \lambda^2 + A_n\lambda + B_n + (C_n - d\lambda)\mathrm{e}^{\lambda\tau} = 0, \tag{5}$$

where

$$\begin{aligned} A_n &= (D_1 + D_2)n^2 + d - \alpha_1, \\ B_n &= D_1 D_2 n^4 + (D_1 d - D_2\alpha_1)n^2 - d\alpha_1, \\ C_n &= -D_1 d n^2 + d\alpha_1 + \alpha_2\beta_1. \end{aligned}$$

Assuming condition $(A)$ as follow:

$$(A) \quad \alpha_1 < 0 \Leftrightarrow r + \frac{adr}{k\mu} < \frac{2adr}{k(\mu - d)} + \frac{r(\mu - d)}{\mu} + \frac{b(\mu - d)^2}{ad\mu},$$

then we have *Lemma 2.1.* If $(A)$ holds, then $\lambda = 0$ is not the root of Eq. (5) for any $n \in \mathbb{N}_0$ .
*Proof.* According to (5) we know that

$$\triangle_n(0, \tau) = B_n + C_n = D_2 n^2(D_1 n^2 - \alpha_1) + \alpha_2\beta_1$$

Condition $(A)$ holds means that $\alpha_1 < 0$, and from (3) we can get $\alpha_2 > 0, \ \ \beta_1 > 0$, then

$$\triangle_n(0, \tau) > 0, \forall\, n \in \mathbb{N}_0.$$

*Lemma 2.2. If $(A)$ holds, when $\tau = 0$, all roots of Eq. (5) have negative real parts,for any $n \in \mathbb{N}_0$, furthermore, $E_*$ is locally asymptotically stable.*
*Proof.* When $\tau = 0$, Eq. (5) becomes

$$\triangle_n(\lambda, 0) = \lambda^2 + (A_n - d)\lambda + B_n + C_n = 0.$$

Suppose that $\lambda_{1n}, \ \lambda_{2n}$ be roots of the above equation, then we have

$$\begin{gathered} \lambda_{1n} + \lambda_{2n} = d - A_n = \alpha_1 - (D_1 + D_2) < 0 \\ \lambda_{1n}\lambda_{2n} = B_n + C_n = D_1 D_2 n^4 - D_2\alpha_1 n^2 + \alpha_2\beta_1 > 0, \end{gathered}$$

which completes the proof.
When $\tau \neq 0$, letting $\imath\omega$ $(\omega > 0)$ be the root of Eq.(5), then

$$\begin{aligned} C_n \sin\omega\tau + d\omega\cos\omega\tau &= A_n\omega \\ C_n \cos\omega\tau - d\omega\sin\omega\tau &= \omega^2 - B_n \end{aligned}$$

which leads to

$$\omega^4 + (A_n^2 - 2B_n - d^2)\omega^2 + B_n^2 - C_n^2 = 0. \tag{6}$$

Let $z = \omega^2$, Eq. (6) becomes

$$z^2 + (A_n^2 - 2B_n - d^2)z + B_n^2 - C_n^2 = 0. \tag{7}$$

Through calculation we know that

$$\begin{aligned} A_n^2 - 2B_n - d^2 &= [(D_1 + D_2)n^2 + d - \alpha_1]^2 - 2[D_1 D_2 n^4 + (D_1 d - D_2\alpha_1)n^2 - d\alpha_1] - d^2, \\ &= D_2 n^4 + 2D_2 dn^2 + (D_1 n^2 - \alpha_1)^2 > 0, \\ B_n^2 - C_n^2 &= (B_n + C_n)(B_n - C_n), \\ B_n + C_n &= D_1 D_2 n^4 - D_2\alpha_1 n^2 + \alpha_2\beta_1, \\ B_n - C_n &= D_1 D_2 n^4 + (2D_1 d - D_2\alpha_1)n^2 - 2\alpha_1 d - \alpha_2\beta_1. \end{aligned}$$

Suppose condition $(A)$ holds. It is obvious that $B_n + C_n > 0$. Furthermore, if $-2\alpha_1 d - \alpha_2\beta_1 > 0$, then $B_n - C_n > 0, \ \forall \ n \in \mathbb{N}_0$; if $-2\alpha_1 d - \alpha_2\beta_1 < 0$, since$B_0 - C_0 = -2\alpha_1 d - \alpha_2\beta_1 < 0$, then there exists $N \in \mathbb{N}$, such that $B_n - C_n < 0$ when $n < N$ and $B_n - C_n > 0$ when $n > N$. So we get that if $(A)$ and $-2\alpha_1 d - \alpha_2\beta_1 > 0$ hold, then Eq. (7) has no positive root for $\forall \ n \in \mathbb{N}_0$ ; and if $(A)$ and $-2\alpha_1 d - \alpha_2\beta_1 < 0$ hold, Eq. (7) has a positive root

$$z_n = \frac{1}{2}[2B_n + d^2 - A_n^2 + \sqrt{(A_n^2 - 2B_n - d^2)^2 - 4(B_n^2 - C_n^2)}]$$

when $n < N$, which means that Eq. (6) also has a positive root

$$\omega_n = \frac{1}{\sqrt{2}}\sqrt{2B_n + d^2 - A_n^2 + \sqrt{(A_n^2 - 2B_n - d^2)^2 - 4(B_n^2 - C_n^2)}}$$

*Lemma 2.3. If $(A)$ and $-2\alpha_1 d - \alpha_2\beta_1 < 0$ hold, then there exist $N \in \mathbb{N}$, such that Eq. (5) has a pair of conjugate pure imaginary roots $\pm \imath\omega_n$ when $\tau = \tau_n^j$ $(n < N, \ j = 0, 1, 2, \cdots)$, where*

$$\tau_n^j = \begin{cases} \dfrac{1}{\omega_n}\left(\arccos\dfrac{a^*}{c^*} + 2j\pi\right), & b^* > 0 \\ \dfrac{1}{\omega_n}\left(2\pi - \arccos\dfrac{a^*}{c^*} + 2j\pi\right), & b^* < 0 \end{cases} \tag{8}$$

here $a^* = (dA_n + C_n)\omega^2 - B_n C_n, \ \ b^* = \omega(A_n C_n - dB_n) - d\omega^3, \ \ c^* = C_n^2 + d^2\omega^2.$

Furthermore, we have *Lemma 2.4.* $\mathrm{Re}(\mathrm{d}\lambda/\mathrm{d}\tau)|_{\tau=\tau_n^j} > 0$

*Proof.* Since (5), we know that

$$[2\lambda + A_n - de^{-\lambda\tau} - \tau(C_n - d\lambda)\lambda e^{-\lambda\tau}]\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = (C_n - d\lambda)\lambda e^{-\lambda\tau},$$

which leads to

$$\begin{aligned} \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} &= \frac{2\lambda + A_n - de^{-\lambda\tau}}{\lambda(C_n - d\lambda)e^{-\lambda\tau}} - \frac{\tau}{\lambda} \\ &= \frac{2\lambda + A_n}{-\lambda(\lambda^2 + A_n\lambda + B_n)} - \frac{d}{\lambda(C_n - d\lambda)} - \frac{\tau}{\lambda}, \end{aligned}$$

substituting $\tau = \tau_n^j$ into the above formula, we have

$$\begin{aligned} \mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_n^j} &= \mathrm{Re}\left[\frac{2\imath\omega_n + A_n}{-\imath\omega_n(-\omega_n^2 + \imath\omega_n A_n + B_n)} - \frac{d}{\imath\omega_n(C_n - \imath d\omega_n)} - \frac{\tau_n^j}{\imath\omega_n}\right] \\ &= \frac{A_n^2 + 2(\omega_n^2 - B_n)}{A_n^2\omega_n^2 + (\omega_n^2 - B_n^2)^2} - \frac{d^2}{d^2\omega_n^2 + C_n^2} \\ &= \frac{A_n^2 - 2B_n - d^2 + 2\omega_n^2}{d^2\omega_n^2 + C_n^2} > 0. \end{aligned}$$

Denoting $\tau_* = \min\{\tau_n^0\}(0 \leq n < N)$, we come to the main conclusion for this section:

*Theorem 2.5.For system (1), suppose that condition $(A)$ holds,*

(i) *If* $-2\alpha_1 d - \alpha_2\beta_1 > 0$*, then* $E_*(u_*, v_*)$ *is locally asymptotically stable for* $\forall\, \tau \in [0, +\infty)$*;*

(ii) *If* $-2\alpha_1 d - \alpha_2\beta_1 < 0$*, then* $E_*(u_*, v_*)$ *is locally asymptotically stable when* $\tau \in [0, \tau_*)$ *and unstable when* $\tau > \tau_*$*; and system undergoes Hopf bifurcation at* $E_*(u_*, v_*)$ *when* $\tau = \tau_n^j (0 \le n < N, j = 0, 1, 2, \cdots)$.

*The bifurcating period solutions are spatially homogenous when* $\tau = \tau_0^j (j = 0, 1, 2, \cdots)$*;*
*The bifurcating period solutions are spatially non-homogenous when* $\tau = \tau_n^j (0 < n < N,\ j = 0, 1, 2, \cdots)$.

# 3. Properties of Hopf Bifurcation

Basing on the discussion in the previous section, we find that under some condition system (1) undergoes Hopf bifurcation, which means that system (1) has bifurcating period solutions. This section, using theory of partial differential equation from Wu [26] and Hassard [27], we study the properties of bifurcating period solutions.

Letting $\bar{u}(t, x) = u(\tau t, x),\ \bar{v}(t, x) = v(\tau t, x)$, and removing the " $-$ " for convenience, expanding system (2) around the origin, we get the following equation:

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \tau D_1 \triangle u(t,x) + \tau[\alpha_1 u(t,x) - \alpha_2 v(t,x) + \alpha_3 u^2(t,x) + \alpha_4 u(t,x)v(t,x) \\ &\quad + \alpha_5 u^3(t,x) + \alpha_6 u^2(t,x)v(t,x) + O(4)] \\ \frac{\partial v(t,x)}{\partial t} &= \tau D_2 \triangle v(t,x) + \tau[-dv(t,x) + \beta_1 u(t-\tau,x) + dv(t-\tau,x) + \beta_2 u^2(t-1,x) \\ &\quad + \beta_3 u(t-1,x)v(t-1,x) + \beta_4 u^3(t-1,x) + \beta_5 u^2(t-1,x)v(t-1,x) + O(4)]\end{aligned} \tag{9}$$

where

$$\begin{aligned}&\alpha_3 = -\frac{r}{k} + \frac{av_*}{(a+u_*)^3}, \quad \alpha_4 = \frac{-a}{(a+u_*)^2}, \quad \alpha_5 = \frac{av_*}{(a+u_*)^4}, \quad \alpha_6 = \frac{-a}{(a+u_*)^3} \\ &\beta_2 = -\frac{a\mu v_*}{(a+u_*)^3}, \qquad \beta_3 = \frac{a\mu}{(a+u_*)^2}, \quad \beta_4 = \frac{a\mu v_*}{(a+u_*)^4}, \quad \beta_5 = \frac{-a\mu}{(a+u_*)^3}\end{aligned}$$

Let

$$u_1(t) = u(t,\cdot),\ v_1(t) = v(t,\cdot),\ U = (u_1, u_2)^T$$

then system (9) can be written as the following abstract ordinary differential equation in the phase space $C = C([-1, 0], X)$

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \bar{\tau} D \triangle U(t) + L_{\bar{\tau}}(U_t) + F(U_t, \varepsilon) \tag{10}$$

Here $L : C \mapsto X,\ \ F : C \mapsto X$ are defined as

$$L_\varepsilon(\phi) = \varepsilon \begin{pmatrix} \alpha_1\phi_1(0) - \alpha_2\phi_2(0) \\ -d\phi_1(0) + \beta_1\phi_1(-1) + d\phi_2(-1) \end{pmatrix}$$

and

$$\begin{gathered} F(\phi, \varepsilon) = \varepsilon D \triangle \phi + L_\varepsilon(\phi) + f(\phi, \varepsilon) f(\phi, \varepsilon) = (\bar{\tau} + \varepsilon) \\ \begin{pmatrix} \alpha_3\phi_1^2(0) + \alpha_4\phi_1(0)\phi_2(0) + \alpha_5\phi_1^3(0) + \alpha_6\phi_1^2(0)\phi_2(0) + O(4) \\ \beta_2\phi_1^2(-1) + \beta_3\phi_1(-1)\phi_2(-1) + \beta_4\phi_1^3(-1) + \beta_5\phi_1^2(-1)\phi_2(-1) + O(4) \end{pmatrix}\end{gathered} \tag{11}$$

where $\phi = (\phi_1, \phi_2)^T \in C$.

Consider the following linear equation

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \bar{\tau} D \triangle U(t) + L_{\bar{\tau}}(U_t) \tag{12}$$

When $\tau = \bar{\tau}$, system (9) has pure imaginary roots, denoted as $\Lambda = \{\mathrm{i}\bar{\omega}\bar{\tau},\ -\mathrm{i}\bar{\omega}\bar{\tau}\}$, what's more, $\Lambda$ are also eigenvalues of the following functional differential equation.

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = -\bar{\tau} D n^2 z(t) + L_{\bar{\tau}}(z_t). \tag{13}$$

By Riesz representation theorem, there exists a $2\times 2$ matrix $\eta^n(\theta,\bar\tau)$ $(-1\le\theta\le 0)$, whose elements are of bounded variation functions, such that

$$-\bar\tau Dn^2\phi(0)+L_{\bar\tau}(\phi)=\int_{-1}^{0}\mathrm{d}\eta^n(\theta,\bar\tau)\phi(\theta).$$

$$\eta^n(\theta,\bar\tau)=\begin{cases}\bar\tau\begin{pmatrix}\alpha_1 & -\alpha_2\\ 0 & 0\end{pmatrix}, & \theta=0\\ 0, & \theta\in(-1,0)\\ \bar\tau\begin{pmatrix}0 & 0\\ \beta_1 & d\end{pmatrix}, & \theta=-1\end{cases}$$

For $\forall\phi\in\mathcal{C}\cap C^1$ and $\psi\in\mathcal{C}\cap C^1$, define the operator $A(\varepsilon)$ and $A^*$ respectively as follow

$$A(\varepsilon)\phi(\theta)=\begin{cases}\dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta\in[-1,0),\\ \int_{-1}^{0}\mathrm{d}\eta(\theta,\varepsilon)\phi(\theta), & \theta=0,\end{cases}$$

$$A^*\psi(s)=\begin{cases}-\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, s\in(0,1],\\ \int_{-1}^{0}\psi(-\xi)\mathrm{d}\eta(\xi,0), s=0.\end{cases}$$

Define the bilinear form

$$\begin{aligned}(\phi,\psi)&=\psi(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\psi(\xi-\theta)\mathrm{d}\eta^n(\theta,\bar\tau)\phi(\xi)\mathrm{d}\xi\\ &=\psi(0)\phi(0)+\bar\tau\int_{-1}^{0}\psi(\xi+1)\begin{pmatrix}0 & 0\\ \beta_1 & d\end{pmatrix}\phi(\xi)\mathrm{d}\xi\end{aligned}\tag{14}$$

where $\phi\in C([-1,0],\mathbb{R}^2)$, $\psi\in C([0,1],\mathbb{R}^*)$. Then $A(0)$ and $A^*$ are adjoint operators, and $\pm\mathrm{i}\bar\omega\bar\tau$ are eigenvalues of $A(\bar\tau)$ and $A^*$. Let $P$ and $P^*$ are center subspaces of $A(\bar\tau)$ and $A^*$ associated with $\Lambda=\{\pm\mathrm{i}\bar\omega\bar\tau\}$ respectively, then $P^*$ is the adjoint space of $P$, and $\mathrm{dim}P=\mathrm{dim}P^*=2$.

By a straightforward computation, we can get that

$$p_1(\theta)=(1,\xi)^T\mathrm{e}^{\mathrm{i}\bar\omega\bar\tau}(\theta\in[-1,0]),\ \ p_2(\theta)=\overline{p_1(\theta)}$$

and

$$q_1(s)=(\eta,1)\mathrm{e}^{-\mathrm{i}\bar\omega\bar\tau}(s\in[0,1]),\ \ q_2(s)=\overline{p_1(s)}$$

are bases of $A(\bar\tau)$ and $A^*$ associated with $\Lambda=\{\pm\mathrm{i}\bar\omega\bar\tau\}$, where

$$\begin{aligned}\xi&=\frac{1}{\alpha_2}(\alpha_1-D_1n^2-\mathrm{i}\bar\omega),\\ \eta&=\frac{1}{\alpha_2}[-d-D_2n^2+d\cos\bar\omega\bar\tau+\mathrm{i}(\bar\omega+d\sin\bar\omega\bar\tau)].\end{aligned}$$

Taking

$$\begin{aligned}\Phi_1(\theta)&=\frac{1}{2}[p_1(\theta)+p_2(\theta)]=\begin{pmatrix}\mathrm{Re}(\mathrm{e}^{\mathrm{i}\bar\omega\bar\tau\theta})\\ \mathrm{Re}(\xi\mathrm{e}^{\mathrm{i}\bar\omega\bar\tau\theta})\end{pmatrix}\\ &=\begin{pmatrix}\cos(\bar\omega\bar\tau\theta)\\ \dfrac{1}{\alpha_2}[(\alpha_1-D_1n^2)\cos(\bar\omega\bar\tau\theta)+\bar\omega\sin(\bar\omega\bar\tau\theta)]\end{pmatrix}\\ \Phi_2(\theta)&=\frac{1}{2}[p_1(\theta)-p_2(\theta)]=\begin{pmatrix}\mathrm{Im}(\mathrm{e}^{\mathrm{i}\bar\omega\bar\tau\theta})\\ \mathrm{Im}(\xi\mathrm{e}^{\mathrm{i}\bar\omega\bar\tau\theta})\end{pmatrix}\\ &=\begin{pmatrix}\sin(\bar\omega\bar\tau\theta)\\ \dfrac{1}{\alpha_2}[(\alpha_1-D_1n^2)\sin(\bar\omega\bar\tau\theta)-\bar\omega\cos(\bar\omega\bar\tau\theta)]\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\Psi_1^*(s) &= \frac{1}{2}[q_1(s)+q_2(s)] = \left(\mathrm{Re}(\eta \mathrm{e}^{-\mathrm{i}\bar\omega\bar\tau\theta}), \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\bar\omega\bar\tau\theta})\right)\\ &= \left(\frac{1}{\alpha_2}[(-d-D_2n^2+d\cos(\bar\omega\bar\tau s))\cos(\bar\omega\bar\tau s)+(\bar\omega+d\sin(\bar\omega\bar\tau s))\sin(\bar\omega\bar\tau s)], \cos(\bar\omega\bar\tau s)\right)\\ \Psi_2^*(s) &= \frac{1}{2}[q_1(s)-q_2(s)] = \left(\mathrm{Im}(\eta \mathrm{e}^{-\mathrm{i}\bar\omega\bar\tau s}), \mathrm{Im}(\mathrm{e}^{-\mathrm{i}\bar\omega\bar\tau s})\right)\\ &= \left(\frac{1}{\alpha_2}[(d+D_2n^2-d\cos(\bar\omega\bar\tau s))\sin(\bar\omega\bar\tau s)+(\bar\omega+d\sin(\bar\omega\bar\tau s))\cos(\bar\omega\bar\tau s)], -\sin(\bar\omega\bar\tau s)\right)\end{aligned}$$

then $(\Phi_1,\Phi_2)$ and $(\Psi_1^*,\Psi_2^*)$ are bases for $P$ and $P^*$ respectively.

By (14), we have

$$\begin{aligned}M_1 &:= (\Psi_1^*,\Phi_1) = \frac{1}{2\alpha_2}[2\gamma+\bar\tau\gamma\rho-d\bar\omega\bar\tau\sin(\bar\omega\bar\tau)]\\ M_2 &:= (\Psi_1^*,\Phi_2) = \frac{-1}{2\alpha_2}[2\bar\omega+d\bar\omega\bar\tau\rho+\bar\tau\gamma\sin(\bar\omega\bar\tau)]\\ M_3 &:= (\Psi_2^*,\Phi_1) = \frac{1}{2\alpha_2}[2\bar\omega-d\bar\omega\bar\tau\delta-\bar\tau\gamma\sin(\bar\omega\bar\tau)]\\ M_4 &:= (\Psi_2^*,\Phi_2) = \frac{1}{2\alpha_2}[\bar\tau\gamma\delta-d\bar\omega\bar\tau\sin(\bar\omega\bar\tau)]\end{aligned}$$

where

$$\gamma=\alpha_1-D_1n^2-D_2n^2,\quad \rho=\cos(\bar\omega\bar\tau)+\frac{\sin(\bar\omega\bar\tau)}{\bar\omega\bar\tau},\quad \delta=\cos(\bar\omega\bar\tau)-\frac{\sin(\bar\omega\bar\tau)}{\bar\omega\bar\tau}.$$

Define

$$(\Psi,\Phi)=(\Psi_j^*,\Phi_k)=\begin{pmatrix}(\Psi_1^*,\Phi_1) & (\Psi_1^*,\Phi_2)\\ (\Psi_2^*,\Phi_1) & (\Psi_2^*,\Phi_2)\end{pmatrix}$$

and choose $\Psi=(\Psi_1,\Psi_2)^T=(\Psi^*,\Phi)^{-1}\Psi^*$, then we have $(\Psi,\Phi)=I_2$, and

$$\begin{aligned}\Psi_1(0)-\mathrm{i}\Psi_2(0) &= \frac{1}{M_1M_4-M_2M_3}\left(\frac{-D_2n^2}{\alpha_2}(M_3+M_4\mathrm{i})-\bar\omega(M_2+M_1\mathrm{i}), M_4+M_3\mathrm{i}\right)\\ &:= (\Upsilon_1,\Upsilon_2)\end{aligned} \tag{15}$$

Take $f_n=(\beta_n^1,\beta_n^2)$, here

$$\beta_n^1=\begin{pmatrix}\cos(nx)\\0\end{pmatrix},\quad \beta_n^2=\begin{pmatrix}0\\ \cos(nx)\end{pmatrix}$$

and denote

$$c\cdot f_n := c_1\beta_n^1+c_2\beta_n^2,\ \ c=(c_1,c_2)^T\in C([-1,0],X)$$

For $u=(u_1,u_2),\ v=(v_1,v_2),\ u,v\in X$, define inner product as

$$\langle u,v\rangle=\frac{1}{\pi}\int_0^\pi u_1\bar v_1\mathrm{d}x+\frac{1}{\pi}\int_0^\pi u_2\bar v_2\mathrm{d}x.$$

For $\phi\in C=C([-1,0],X)$, define $\langle\phi,f_n\rangle=(\langle\phi,f_n^1\rangle,\langle\phi,f_n^2\rangle)^T$, then we get the center subspace of Eq. (12)

$$P_{CN}C(\phi)=\Phi(\Psi,\langle\phi,f_k\rangle)\cdot f_k,\ \ \phi\in C,$$

and we can decompose space into $C=P_{CN}C\oplus P_SC$, where $P_SC$ is the complementary space of $P_{CN}C$ in $C$.

Let $A_{\bar\tau}$ be the infinitesimal generator of the semigroup induced by solutions of Eq. (13), then (2) can be written as the abstract ODE

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t}=A_{\bar\tau}U_t+R(U_t,\varepsilon) \tag{16}$$

where

$$R(U_t,\varepsilon)=\begin{cases}0, & \theta\in[-1,0)\\ F(U_t,\varepsilon), & \theta=0\end{cases}$$

We can decompose the abstract equation into

$$U_t = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot f_n + h(x_1, x_2, \varepsilon),$$

where

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (\Psi, \langle U_t, f_n \rangle), \quad h(x_1, x_2, \varepsilon) \in P_S C, \quad h(0,0,0) = 0, \quad Dh(0,0,0) = 0.$$

Solutions of (10) on the center manifold are

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_n + h(x_1, x_2, 0) \tag{17}$$

Let $z(t) = x_1(t) - \mathrm{i}x_2(t)$, then we have

$$\dot{z}(t) = \mathrm{i}\bar{\omega}\bar{\tau} z(t) + g(z, \bar{z})$$

where

$$g(z, \bar{z}) = (\Psi_1(0) - \mathrm{i}\Psi_2(0)) \langle F(U_t, 0), f_n \rangle$$

Substitute $z(t)$ into (17), we get

$$\Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_n = (\Phi_1, \Phi_2) \begin{pmatrix} \dfrac{z + \bar{z}}{2} \\ \dfrac{\mathrm{i}(z - \bar{z})}{2} \end{pmatrix} \cdot f_n = \frac{1}{2}(p_1 z + \bar{p}_1 \bar{z}) \cdot f_n$$

and

$$h(x_1, x_2, 0) = h\left(\frac{z + \bar{z}}{2}, \frac{\mathrm{i}(z - \bar{z})}{2}, 0\right)$$

then $U_t$ can be changed into the following form

$$U_t = \frac{1}{2}(p_1 z + \bar{p}_1 \bar{z}) \cdot f_n + h\left(\frac{z + \bar{z}}{2}, \frac{\mathrm{i}(z - \bar{z})}{2}, 0\right) := \frac{1}{2}(p_1 z + \bar{p}_1 \bar{z}) \cdot f_n + W(z, \bar{z})$$

Let

$$W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11} z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots, \quad g(z, \bar{z}) \quad = g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots \tag{18}$$

then by (11) and (18) we get

$$\begin{aligned}
\langle F(U_t, 0), f_n \rangle = {} & \frac{\bar{\tau} z^2}{4} \begin{pmatrix} \alpha_3 + \alpha_4 \xi \\ \mathrm{e}^{-2\mathrm{i}\bar{\omega}\bar{\tau}}(\beta_2 + \beta_3 \xi) \end{pmatrix} \frac{1}{\pi} \int_0^\pi \cos^3(nx)\mathrm{d}x \\
& + \frac{\bar{\tau} z\bar{z}}{4} \begin{pmatrix} 2\alpha_3 + \alpha_4(\xi + \bar{\xi}) \\ 2\beta_2 + \beta_3(\xi + \bar{\xi}) \end{pmatrix} \frac{1}{\pi} \int_0^\pi \cos^3(nx)\mathrm{d}x \\
& + \frac{\bar{\tau} \bar{z}^2}{4} \begin{pmatrix} \alpha_3 + \alpha_4 \bar{\xi} \\ \mathrm{e}^{2\mathrm{i}\bar{\omega}\bar{\tau}}(\beta_2 + \beta_3 \bar{\xi}) \end{pmatrix} \frac{1}{\pi} \int_0^\pi \cos^3(nx)\mathrm{d}x \\
& + \bar{\tau} z^2 \bar{z} \begin{pmatrix} \begin{aligned} & \alpha_3 \left\langle \left( W_{11}^{(1)}(0) + \frac{1}{2} W_{20}^{(1)}(0) \right) \cos(nx), \cos(nx) \right\rangle \\ & + \alpha_4 \left\langle \left( \frac{1}{2} W_{11}^{(2)}(0) + \frac{1}{4} W_{20}^{(2)}(0) + \frac{\bar{\xi}}{4} W_{20}^{(1)}(0) + \frac{\xi}{2} W_{11}^{(1)}(0) \right) \cos(nx), \cos(nx) \right\rangle \\ & + \frac{1}{8}(3\alpha_3 + \alpha_6 \bar{\xi} + 2\alpha_6 \xi) \langle \cos^3(nx), \cos(nx) \rangle \\ & \beta_2 \left\langle \left( \mathrm{e}^{-\mathrm{i}\bar{\omega}\bar{\tau}} W_{11}^{(1)}(-1) + \frac{\mathrm{e}^{\mathrm{i}\bar{\omega}\bar{\tau}}}{2} W_{20}^{(1)}(-1) \right) \cos(nx), \cos(nx) \right\rangle \\ & + \beta_3 \left\langle \left( \frac{\mathrm{e}^{-\mathrm{i}\bar{\omega}\bar{\tau}}}{2} (W_{11}^{(2)}(-1) + \xi W_{11}^{(1)}(-1)) \right) \cos(nx), \cos(nx) \right\rangle \\ & + \beta_3 \left\langle \left( \frac{\mathrm{e}^{\mathrm{i}\bar{\omega}\bar{\tau}}}{4} (W_{20}^{(2)}(-1) + \bar{\xi} W_{20}^{(1)}(-1)) \right) \cos(nx), \cos(nx) \right\rangle \\ & + \frac{\mathrm{e}^{-\mathrm{i}\bar{\omega}\bar{\tau}}}{8}(3\beta_4 + \beta_5 \bar{\xi} + 2\beta_5 \xi) \langle \cos^3(nx), \cos(nx) \rangle \end{aligned} \end{pmatrix}
\end{aligned}$$

Since $\int_0^\pi \cos^3(nx)\mathrm{d}x = 0$ when$n \neq 0$ , combining with (15), we have

$$g_{20} = \overline{g_{02}} = \begin{cases} 0, & n \neq 0 \\ \dfrac{\bar{\tau}}{2}[\Upsilon_1(\alpha_3 + \xi\alpha_4) + \Upsilon_2 \mathrm{e}^{-2\mathrm{i}\bar{\omega}\bar{\tau}}(\beta_2 + \xi\beta_3)], & n = 0 \end{cases}$$

$$g_{11} = \begin{cases} 0, & n \neq 0 \\ \dfrac{\bar{\tau}}{4}[\Upsilon_1(2\alpha_3 + (\xi + \bar{\xi})\alpha_4) + \Upsilon_2(2\beta_2 + (\xi + \bar{\xi})\beta_3)], & n = 0 \end{cases}$$

$$g_{21} = \bar{\tau}\Upsilon_1 \begin{pmatrix} \alpha_3 \left\langle \left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right)\cos(nx), \cos(nx) \right\rangle \\ + \alpha_4 \left\langle \left(W_{11}^{(2)}(0) + \dfrac{1}{2}W_{20}^{(2)}(0) + \dfrac{\bar{\xi}}{2}W_{20}^{(1)}(0) + \xi W_{11}^{(1)}(0)\right)\cos(nx), \cos(nx) \right\rangle \\ + \dfrac{1}{4}(3\alpha_3 + \alpha_6\bar{\xi} + 2\alpha_6\xi)\langle \cos^3(nx), \cos(nx) \rangle \end{pmatrix}$$

$$+ \bar{\tau}\Upsilon_2 \begin{pmatrix} \beta_2 \left\langle \left(2\mathrm{e}^{-\mathrm{i}\bar{\omega}\bar{\tau}}W_{11}^{(1)}(-1) + \mathrm{e}^{\mathrm{i}\bar{\omega}\bar{\tau}}W_{20}^{(1)}(-1)\right)\cos(nx), \cos(nx) \right\rangle \\ + \beta_3 \left\langle \left(\mathrm{e}^{-\mathrm{i}\bar{\omega}\bar{\tau}}(W_{11}^{(2)}(-1) + \xi W_{11}^{(1)}(-1))\right)\cos(nx), \cos(nx) \right\rangle \\ + \beta_3 \left\langle \left(\dfrac{\mathrm{e}^{\mathrm{i}\bar{\omega}\bar{\tau}}}{2}(W_{20}^{(2)}(-1) + \bar{\xi}W_{20}^{(1)}(-1))\right)\cos(nx), \cos(nx) \right\rangle \\ + \dfrac{\mathrm{e}^{-\mathrm{i}\bar{\omega}\bar{\tau}}}{4}(3\beta_4 + \beta_5\bar{\xi} + 2\beta_5\xi)\langle \cos^3(nx), \cos(nx) \rangle \end{pmatrix}$$

Here we need to compute $W_{20}(\theta),\ W_{11}(\theta),\ \ \theta \in [-1, 0]$.
By (18)

$$\begin{aligned} \dot{W}(z, \bar{z}) &= W_{20}z\dot{z} + W_{11}\dot{z}\bar{z} + W_{11}z\dot{\bar{z}} + W_{02}\bar{z}\dot{\bar{z}} + \cdots \\ &:= A_{\bar{\tau}}W + H(z, \bar{z}) \\ A_{\bar{\tau}}W(z, \bar{z}) &= A_{\bar{\tau}}W_{20}\frac{z^2}{2} + A_{\bar{\tau}}W_{11}z\bar{z} + A_{\bar{\tau}}W_{02}\frac{\bar{z}^2}{2} + \cdots \end{aligned}$$

here

$$\begin{aligned} H(z, \bar{z}) &= H_{20}\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots \\ &= X_0F(U_t, 0) - \Phi(\Psi, \langle X_0F(U_t, 0), f_n \rangle) \cdot f_n \end{aligned}$$

so

$$(2\mathrm{i}\bar{\omega}\bar{\tau} - A_{\bar{\tau}})W_{20} = H_{20}, \quad -A_{\bar{\tau}}W_{11} = H_{11}, \ \ (-2\mathrm{i}\bar{\omega}\bar{\tau} - A_{\bar{\tau}})W_{02} = H_{02}, \tag{19}$$

here we need to find out $H_{20}(\theta)$ and $H_{11}(\theta)$.
When $\theta \in [-1, 0)$, we know that

$$\begin{aligned} H(z, \bar{z}) &= -\Phi(\theta)\Psi(0)\langle F(U_t, 0), f_n \rangle \cdot f_n \\ &= -\left(\frac{p_1(\theta) + p_2(\theta)}{2}, \frac{p_1(\theta) - p_2(\theta)}{2\mathrm{i}}\right)\begin{pmatrix} \Psi_1(0) \\ \Psi_2(0) \end{pmatrix}\langle F(U_t, 0), f_n \rangle \cdot f_n \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{2}[p_1(\theta)(\Psi_1(0) - \mathrm{i}\Psi_2(0)) + p_2(\theta)(\Psi_(0) + \mathrm{i}\Psi_2(0))]\langle F(U_t, 0), f_n \rangle \cdot f_n \\ &= -\frac{1}{2}(p_1(\theta)g_{20} + p_2(\theta)\bar{g}_{02}) \cdot f_n\frac{z^2}{2} - \frac{1}{2}(p_1(\theta)g_{11} + p_2(\theta)\bar{g}_{11}) \cdot f_n\frac{z^2}{2} + \cdots \\ &\quad - \frac{1}{2}(p_1(\theta)g_{02} + p_2(\theta)\bar{g}_{20}) \cdot f_n\frac{\bar{z}^2}{2} \end{aligned}$$

so we have

$$H_{20}(\theta) = \begin{cases} 0, & n = 1, 2, 3, \cdots \\ -\frac{1}{2}(p_1(\theta)g_{20} + p_2(\theta)\bar{g}_{02}) \cdot f_0, & n = 0 \end{cases}$$

$$H_{11}(\theta) = \begin{cases} 0, & n = 1, 2, 3, \cdots \\ -\frac{1}{2}(p_1(\theta)g_{11} + p_2(\theta)\bar{g}_{11}) \cdot f_0, & n = 0 \end{cases}$$

$$H_{02}(\theta) = \begin{cases} 0, & n = 1, 2, 3, \cdots \\ -\frac{1}{2}(p_1(\theta)g_{02} + p_2(\theta)\bar{g}_{20}) \cdot f_0, & n = 0 \end{cases}$$

when $\theta = 0$, $H(z,\bar{z})(0) = F(U_t,0) - \Phi(\Psi, \langle F(U_t,0), f_n\rangle)\cdot f_n$, so we have

$$H_{20}(0) = \begin{cases} \dfrac{\bar{\tau}}{2}\begin{pmatrix} \alpha_3+\alpha_4\xi \\ \mathrm{e}^{-2\mathrm{i}\bar{\omega}\bar{\tau}}(\beta_2+\beta_3\xi) \end{pmatrix}\cos^2(nx), & n = 1,2,3,\cdots \\ \dfrac{\bar{\tau}}{2}\begin{pmatrix} \alpha_3+\alpha_4\xi \\ \mathrm{e}^{-2\mathrm{i}\bar{\omega}\bar{\tau}}(\beta_2+\beta_3\xi) \end{pmatrix} - \dfrac{1}{2}(p_1(\theta)g_{20}+p_2(\theta)\bar{g}_{02})\cdot f_0, & n=0 \end{cases}$$

$$H_{11}(0) = \begin{cases} \dfrac{\bar{\tau}}{4}\begin{pmatrix} 2\alpha_3+\alpha_4(\xi+\bar{\xi}) \\ 2\beta_2+\beta_3(\xi+\bar{\xi}) \end{pmatrix}\cos^2(nx), & n = 1,2,3,\cdots \\ \dfrac{\bar{\tau}}{4}\begin{pmatrix} 2\alpha_3+\alpha_4(\xi+\bar{\xi}) \\ 2\beta_2+\beta_3(\xi+\bar{\xi}) \end{pmatrix} - \dfrac{1}{2}(p_1(\theta)g_{11}+p_2(\theta)\bar{g}_{11})\cdot f_0, & n=0 \end{cases}$$

By the definition of $A_{\bar{\tau}}$ and (19),we have

$$\dot{W}_{20} = A_{\bar{\tau}}W_{20} = 2\mathrm{i}\bar{\omega}\bar{\tau}W_{20} + \frac{1}{2}(p_1g_{20}+p_2\bar{g}_{02})\cdot f_n, \ \ -1\le\theta<0$$

therefore

$$W_{20}(\theta) = \frac{\mathrm{i}}{2\bar{\omega}\bar{\tau}}\Big(g_{20}p_1(\theta)+\frac{\bar{g}_{02}}{3}p_2(\theta)\Big)\cdot f_n + E_1\mathrm{e}^{2\mathrm{i}\bar{\omega}\bar{\tau}\theta}$$

where

$$E_1 = \begin{cases} W_{20}(0), & n=1,2,3,\cdots \\ W_{20}(0) - \dfrac{\mathrm{i}}{2\bar{\omega}\bar{\tau}}\Big(g_{20}p_1(0)+\dfrac{\bar{g}_{02}}{3}p_2(0)\Big)\cdot f_0, & n=0 \end{cases}$$

furthermore, by the definition of $A_{\bar{\tau}}$, we have

$$\begin{aligned} &2\mathrm{i}\bar{\omega}\bar{\tau}\left[\frac{\mathrm{i}}{2\bar{\omega}\bar{\tau}}\Big(g_{20}p_1(0)+\frac{\bar{g}_{02}}{3}p_2(0)\Big)\cdot f_0 + E_1\right] - \bar{\tau}D\triangle\left[\frac{\mathrm{i}}{2\bar{\omega}\bar{\tau}}\Big(g_{20}p_1(0)+\frac{\bar{g}_{02}}{3}p_2(0)\Big)\cdot f_0 + E_1\right] \\ &\quad - L(\bar{\tau})\left[\frac{\mathrm{i}}{2\bar{\omega}\bar{\tau}}\Big(g_{20}p_1(\theta)+\frac{\bar{g}_{02}}{3}p_2(\theta)\Big)\cdot f_0 + E_1\mathrm{e}^{2\mathrm{i}\bar{\omega}\bar{\tau}}\right] \\ &= \frac{\bar{\tau}}{2}\begin{pmatrix} \alpha_3+\alpha_4\xi \\ \mathrm{e}^{-2\mathrm{i}\bar{\omega}\bar{\tau}}(\beta_2+\beta_3\xi) \end{pmatrix} - \frac{1}{2}(p_1(\theta)g_{20}+p_2(\theta)\bar{g}_{02})\cdot f_0 \end{aligned}$$

notice that

$$\begin{aligned} \bar{\tau}D\triangle[p_1(0)\cdot f_0] + L(\bar{\tau})[p_1(\theta)\cdot f_0] &= \mathrm{i}\bar{\omega}\bar{\tau}p_1(0)\cdot f_0 \\ \bar{\tau}D\triangle[p_2(0)\cdot f_0] + L(\bar{\tau})[p_2(\theta)\cdot f_0] &= -\mathrm{i}\bar{\omega}\bar{\tau}p_2(0)\cdot f_0 \end{aligned}$$

we get

$$2\mathrm{i}\bar{\omega}\bar{\tau}E_1 - \bar{\tau}D\triangle E_1 - L(\bar{\tau})(E_1\mathrm{e}^{2\mathrm{i}\bar{\omega}\bar{\tau}\theta}) = \frac{\bar{\tau}}{2}\begin{pmatrix} \alpha_3+\alpha_4\xi \\ \mathrm{e}^{-2\mathrm{i}\bar{\omega}\bar{\tau}}(\beta_2+\beta_3\xi) \end{pmatrix}\cos^2(nx)$$

and

$$E_1 = \begin{pmatrix} 2\mathrm{i}\bar{\omega}+D_1n^2-\alpha_1 & \alpha_2 \\ -\beta_1\mathrm{e}^{-2\mathrm{i}\bar{\omega}\bar{\tau}} & 2\mathrm{i}\bar{\omega}+D_2n^2+d-d\mathrm{e}^{-2\mathrm{i}\bar{\omega}\bar{\tau}} \end{pmatrix}^{-1}\begin{pmatrix} \alpha_3+\alpha_4\xi \\ \mathrm{e}^{-2\mathrm{i}\bar{\omega}\bar{\tau}}(\beta_2+\beta_3\xi) \end{pmatrix}\frac{\cos^2(nx)}{2}$$

Since

$$\dot{W}_{11}(\theta) = \frac{1}{2}[p_1(\theta)g_{11}+p_2(\theta)\bar{g}_{11}]\cdot f_n,$$

then

$$W_{11}(\theta) = \frac{\mathrm{i}}{2\bar{\omega}\bar{\tau}}\left(p_2(\theta)\bar{g}_{11}-p_1(\theta)g_{11}\right) + E_2$$

similarly we get

$$E_2 = \begin{pmatrix} D_1n^2-\alpha_1 & \alpha_2 \\ -\beta_1 & D_2n^2 \end{pmatrix}^{-1}\begin{pmatrix} 2\alpha_3+\alpha_4(\xi+\bar{\xi}) \\ 2\beta_2+\beta_3(\xi+\bar{\xi}) \end{pmatrix}\frac{\cos^2(nx)}{4}.$$

Therefor $g_{21}$ can be expressed by the parameters in system (1). Thus, we can compute the following quantities:

$$\begin{aligned} c_1(0) &= \frac{\mathrm{i}}{2\bar{\omega}\bar{\tau}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2} \\ \mu_2 &= -\frac{\mathrm{Re}c_1(0)}{\mathrm{Re}\lambda'(\bar{\tau})} \\ \beta_2 &= 2\mathrm{Re}c_1(0) \\ T_2 &= -\frac{\mathrm{Im}c_1(0) + \mu_2 \mathrm{Im}\lambda'(\bar{\tau})}{\bar{\omega}\bar{\tau}} \end{aligned}$$

which determine the properties of bifurcating periodic solutions at the critical value $\tau_0$. The direction and stability of Hopf bifurcation in the center manifold can be determined by $\mu_2$ and $\beta_2$ respectively. In fact, if $\mu_2 > 0 (\mu_2 < 0)$, then the bifurcating periodic solutions are forward (backward); the bifurcating periodic solutions on the center manifold are stable (unstable) if $\beta_2 < 0 (\beta_2 > 0)$; and $T_2$ determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0 (T_2 < 0)$.

# 4. Numerical Simulations

We choose a set of parameters as follows:

$$\text{(a)}\ \ a = 1,\ \ b = 0.4,\ d = 0.4, k = 4,\\ r = 6, \mu = 2,\ D_1 = 2,\ \ D_2 = 0.5$$

then $E_0 = (4.06559, 0)$, $E_* = (0.25, 9.03125)$. Basing on the analysis previously, we get

$$B_0 - C_0 < 0,\ \ B_n - C_n > 0\ (n \geq 1)$$

which means that Hopf bifurcation occurs only when $n = 0$ and $\tau_0^0 = 0.2727$.

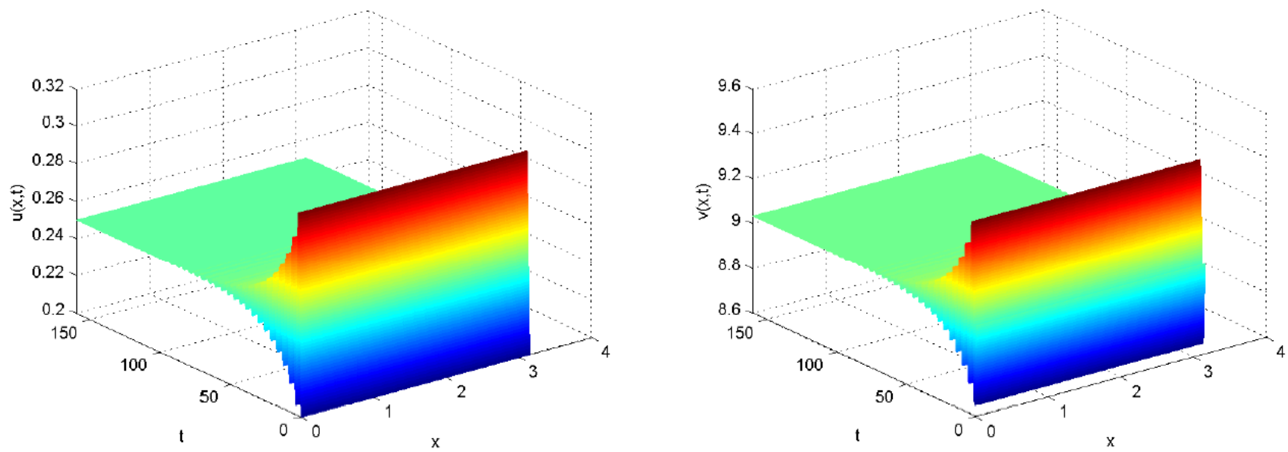


***Figure 1.*** *When $\tau = 0.25 < \tau_0$, the positive equilibrium of system (1) is asymptotically stable with parameters given in* (a)*, and the initial value is* $(0.3, 9)$.

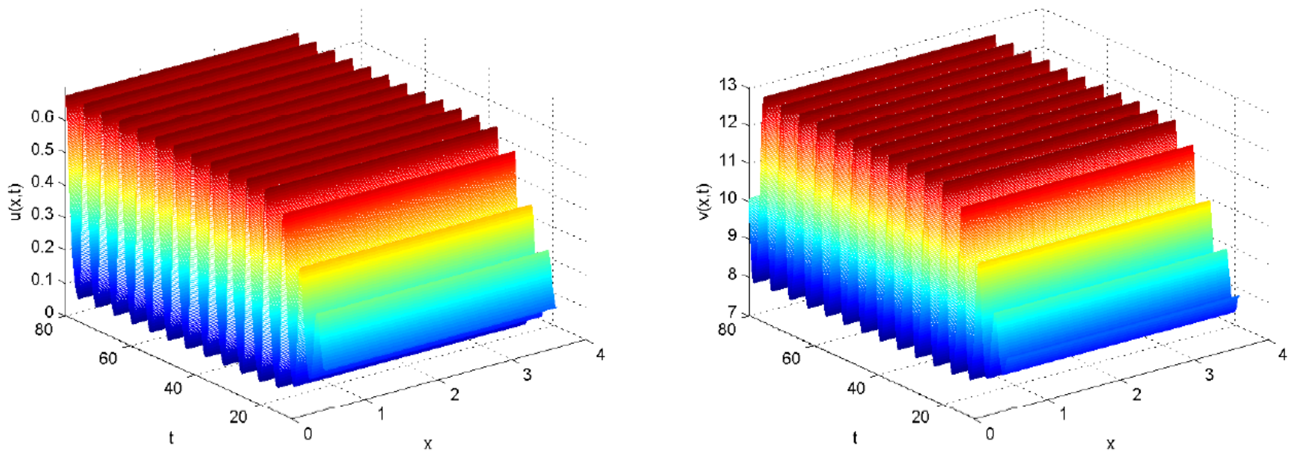


***Figure 2.*** *When $\tau = 0.4 > \tau_0$, the spatial homogeneous periodic solution bifurcating from the positive equilibrium of system (1) with parameters given in* (a)*, and the initial value is* $(0.3, 9)$.

# 5. Conclusion

In this paper, we consider the diffusive Rosenzweig-MacArthur model with a constant delay $\tau$ due to the gestation of predator. We find that if condition $(\mathrm{A})$ holds, then stability of the positive equilibrium $E_*(u_*, v_*)$ was influenced by the choose of the parameters. With the parameter set $(a)$ as shown in section 4, we find that $E_*(u_*, v_*)$ loses its stability and an spatial homogenous orbitally asymptotically stable periodic solution arises from the Hopf bifurcation when the delay $\tau$ passes through some critical value $\tau_0^0$. This shows the important influence of the time delay $\tau$ to the system.

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