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A Strategy for Solving the Non Symmetries Arising in Nonlinear Consolidation of Partially Saturated Soils

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Abstract: The main scope of this paper is to present an alternative to tackle the problem of the non symmetries arising in the solution of the nonlinear couple consolidation problem based on a combination of different stress states. Being originally a non symmetric problem, it may be straightforward reduced to a symmetric one, and the conditions in which this reduction may be carried out, are addressed. Non linear saturation-suction and permeability-suction functions were regarded. The geometric model was developed considering an updated lagrangian description with a co-rotated Kirchhoff stress tensor. This description leads to a non-symmetric stiffness matrix and a simple alternative, using a symmetric constitutive matrix, is addressed to overcome this situation. The whole equation system was solved using an open finite element code FECCUND, developed by the authors. In order to validate the model, various examples, for which previous solutions are known, were solved. The use of either a strongly non linear and no symmetric formulation or a simple symmetric formulation with accurate prediction in deformation and pore-pressures is extremely dependent on the soil characteristic curves and on the shear efforts level, as well. A numerical example show the predictive capability of this geometrically non linear fully coupled model for attaining the proposed goal.

Keywords: Finite Element Analysis, Hypoelastic Formulations, Non Saturated Soil Model, Saturation-Suction Relationship Introduction

1. Introduction

Soil consolidation research is commonly carried out by the application of the finite element method to specific mathematical models. These models have undergone a continuous evolution during the last years, so that important achievements towards the prediction of porous media behavior were attained, based on a robust mathematical framework [15, 17, 24].

From Biot [2] pioneer work, many approaches for consolidation analysis have been made. For the non saturated case, Ghaboussi et al [5] presented a two-phase model; Lewis et al [12] and Pietraszezak et al [19] developed some of the earlier three phase models, and, Ng and Small [18] tackled ordinary problems using similar methods. The non isothermal analysis in saturated models is included in Masters et al [14] among others.

More recently and based on Hassanizadeh and Gray [6, 7, 8], environmental geomechanics topics were faced by Schrefler [21], among some others. Klubertantz et al [10] worked on models with miscible and immiscible pore fluids, addressing domains of applicability for each case.

Mroginski et al. [16] described a kind of odd relationship that bonds the vertical displacements and the degree of pollutant saturation.

Different degrees of heterogeneity were considered in [1] as well as in [23]. In the former, a horizontal multilayered soil with anisotropic permeability undergoing square load was analyzed whilst in the later a first order homogenization on RVE was adopted to overcome the heterogeneous issue. In [20], the physical interpretation of the three characteristic behaviors of homogenized dual-porosity is evaluated along with memory effects.

Regarding with the mathematical framework of the aforementioned models, one controversial topic is the degree of saturation as the main coupling element between water – air fields [9] and the induced matric suction variation. From the present review, it comes up that the suction change gives to the governing equations a highly non-linear characteristic and lead to the loss of symmetry in the isothermal case.

Khalili and Khabbaz [9] presented a mathematical approach for isothermal partially saturated media relied on a stress state decomposition though regardless the saturation and the induced matric suction coupling effect. For this issue was subject of large controversial in Di Rado et al. [4] the evidence of the highly non linear effect that saturation-suction coupling effect renders to the constitutive model and its influence on the symmetry loss of the main system equation for the isothermal case, were properly settled down. Besides, one noteworthy feature of the model is that it may be used to discern when the additional costs in terms of processing time and computational memory due to the lost of symmetry is justified and when not, relied on a consistent mathematical background.

With respect to the geometric description, there are many approaches toward the numerical solution of geometrical nonlinear problems. However, any approach must preserve the mechanical principles and the energy requirement. Here, it was assumed that there is not a strain energy function and, hence, a rate independent hypoelastic co-rotated Kirchhoff stress with large strains and displacement model, was adopted. Interest of this formulation remains [13] because it offers a straightforward extension of the small strain plasticity theory, it is not restricted to isotropic materials and meets the frame indifference principle [22]. In contrast, this kind of model leads to a somewhat awkward implementation due to the appearance of non symmetric matrices and deformation-dependent constitutive tensors. Therefore, a proposal to overcome the drawback due to the loss of symmetry in the stiffness matrix is addressed in [3].

2. Partially Saturated Soil Governing Equations

The present approach, may be depicted as a three element set, namely, a deformation model, a water flux model and an air flux model. A whole description was addressed in [4].

The coupling between the flow and the deformation fields is established by means of the introduction of parameters that connect the water and air phase pressures to the change in the deformation matrix.

From reference [4], the following equations may be regarded:

2.1. Mechanical Equilibrium

Using the Cauchy rate tensor, the stress equilibrium is given by:

$$\dot{\sigma}'_{ij} = \dot{\sigma}_{ij} - a_1 \dot{p}^w \delta_{ij} - a_2 \dot{p}^g \delta_{ij} \quad (1)$$

with $a_1 = (c_m - c_s)/c$ and $a_2 = 1 - c_m/c$, being $c_m = 1/K_m$ the compressibility of soil structure with respect to a change in matric suction $p^c = p^g - p^w$, $c = 1/K_T$ is the drained compressibility of the soil structure. $c_s = 1/K_s$ is the compressibility of the soil grains. $K_T = (1-\alpha)K_s$ is the bulk modulus of the overall skeleton with α being the Biot constant. $K_m = K_T K_s / (S_w K_s + S_g K_T)$ is the soil structure compressibility with respect to matric suction.

Moreover, $\dot{\sigma}'_{ij}$ is the effective stress rate, $\dot{\sigma}_{ij}$ is the total stress rate, $\dot{p}^w$ and $\dot{p}^g$ are the water and gas pressure rate respectively, S_w and S_g the gas and water phase saturation respectively. Also consider that $S_w = V_w/V_v$ and $S_g = V_g/V_v$ where V_w is the pore-water volume, V_g is the pore-air volume and V_v, is the void volume.

Considering a non restricted stress-strain relationship, yields:

$$\dot{\sigma}'_{ij} = C_{ijkl} \dot{\varepsilon}_{kl} \quad (2)$$

The constitutive tensor C_{ijkl} should be elastic or elastoplastic with the limitations pointed out in section 1. The equilibrium equations in rate form are given by:

$$\left(\dot{\sigma}_{ij}\right)_{,j} + \dot{F}_i = 0 \quad (3)$$

Combining Eqs. (1), (2) and (3), the following equation governing the soil deformation is obtained:

$$\left(C_{ijkl}\dot{\varepsilon}_{kl}\right)_{,j} + a_1 \dot{p}^w_{,i} + a_2 \dot{p}^g_{,i} + \dot{F}_i = 0 \quad (4)$$

2.2. Water Phase

Regarding the water phase, the next equation holds

$$\frac{1}{\rho^w}\left(\rho^w \frac{k_w}{\gamma_w} p^w_{,i}\right)_{,i} + \alpha S_w \dot{\varepsilon}_{ii} - \left\{\frac{n S_w}{K_w}\left[\frac{\alpha-n}{K_s} S_w \left(S_w + (dp^g - dp^w)\frac{dS_w}{dp^c}\right) + \frac{\alpha-n}{K_T} S_w \left(S_g - dp^c \frac{dS_w}{dp^c}\right)\right]\right\}\dot{p}^w$$

$$-\left\{\frac{\alpha-n}{K_s} S_w \left[S_g - (dp^g - dp^w)\frac{dS_w}{dp^c}\right] - \frac{\alpha-n}{K_T} S_w \left(S_g - dp^c \frac{dS_w}{dp^c}\right)\right\}\dot{p}^g \quad (5)$$

where k_w is the coefficient of permeability, n is the porosity, γ_w is the water specific weight, ρ^w is the water density.

2.3. Air Phase

Carrying out similar modifications than those performed in the water phase and adding the relationship $\dot{S}_g = \dot{p}^c \, dS_g/dp^c = -\dot{p}^c \, dS_w/dp^c$ the follows:

$$\frac{1}{\rho^g}\left(D^* p^g_{,i}\right)_{,i} + \alpha S_g \dot{\varepsilon}_{ii} - \left\{\frac{\alpha-n}{K_s} S_w S_g + S_g \left[\frac{\alpha-n}{K_s}(dp^g - dp^w)\frac{dS_w}{dp^c} - \frac{\alpha-n}{K_T}\left(S_w - dp^c \frac{dS_w}{dp^c}\right)\right]\right\}\dot{p}^w$$

$$-\left\{\frac{nS_g}{P}+\frac{\alpha-n}{K_s}S_g^2-S_g\left[\frac{\alpha-n}{K_s}(dp^g-dp^w)\frac{dS_w}{dp^c}-\frac{\alpha-n}{K_T}\left(S_w-dp^c\frac{dS_w}{dp^c}\right)\right]\right\}\dot{p}^g \quad (6)$$

in which D^* is the air phase transmission coefficient, ρ^g is the air density, P is the absolute pressure.

2.4. Coupling of the Mechanical Equilibrium with the Fluid Phases

Gathering Eqs. (4), (5) and (6) it is obtained:

$$\left(C_{ijkl}\dot{\varepsilon}_{kl}\right)_{,j}+a_1\dot{p}^w_{,i}+a_2\dot{p}^g_{,i}+\dot{F}_i=0$$

$$a_1\dot{\varepsilon}_{ii}-a_{11}\dot{p}^w-a_{12}\dot{p}^g+\frac{1}{\rho^w}\left(\rho^w\frac{k_w}{\gamma_w}p^w_{,i}\right)_{,i}=0 \quad (7)$$

$$a_2\dot{\varepsilon}_{ii}-a_{21}\dot{p}^w-a_{22}\dot{p}^g+\frac{1}{\rho^g}\left(D^*p^g_{,i}\right)_{,i}=0$$

Equations (7) stands for a system of partial differential equations for the solution of the isothermal consolidation problem obtained by a combination of several stress situations applied on a soil system. This formulation leads to non symmetrical matrices when the finite elements method is applied and may be straightforward reduced to a formulation with symmetric matrices adjusting only the saturation indicator [4].

3. The Geometrical Model

In Eqs. (7), C_{ijkl}, stands for an elastic constitutive tensor. When geometric non linearity is under consideration, this tensor must be replaced for a non linear one. In the following sections, this situation is addressed.

3.1. The Weak Form of the Equilibrium Equation for Dry Soil

In reference [3], the following equation for the weak form of the equilibrium equations, was conveyed:

$$\int_{\varphi(B)}(L_v\,\tau+\mathrm{L}\,\tau):\nabla\mathrm{v}\,d\varphi(B)/J=\int_{\varphi(B)}\dot{\mathrm{b}}v\rho\rho\;\varphi(B)+\int_{\partial\varphi(B)}\dot{\mathrm{t}}\,\mathrm{v}\,d\partial\varphi(B) \quad (8)$$

where $\dot{\mathrm{b}}$ and $\dot{\mathrm{t}}$ are the rate of change of body and traction forces, respectively, at the configuration $\varphi(B)$ with boundary $\partial\varphi(B)$, τ is Kirchhoff stress tensor, $\mathrm{L}\equiv\nabla\mathrm{v}=\partial\mathrm{v}/\partial\mathrm{x}$ is the spatial velocity gradient and $L_v\tau$ is the Lie derivative of Kirchhoff stress tensor. Introducing in Eq. (8) the forthcoming constitutive equation:

$$L_v\tau_{ij}=\mathrm{C}^{\tau}_{ijkl}\mathrm{D}_{kl} \quad (9)$$

being C^{τ} the corresponding associated constitutive tensor and D the spatial rate of deformation tensor, leads to:

$$\int_{\varphi(B)}\delta\mathrm{L}:(\mathrm{L}\,\tau+\mathrm{C}^{\tau}:\mathrm{D})\,d\varphi(B)/J=\int_{\varphi(B)}\delta\mathrm{v}\,\dot{\mathrm{b}}\rho\;d\varphi(B)+\int_{\partial\varphi(B)}\delta\mathrm{v}\,\dot{\mathrm{t}}\,d\partial\varphi(B) \quad (10)$$

3.2. Hypoelastic Material Description

Hypoelastic descriptions of constitutive relationships should be conveniently adopted in order to preserve the material frame indifference principle and, when it is required, anisotropic characteristics. If the elastic response is carried out in the spatial or deformed configuration, using Eq. (10) with a constant constitutive tensor, the objectivity requires C^{τ} being invariant under a rigid body motion of the reference configuration [22] and only isotropic materials meet this requirement. To overcome this restriction the constitutive relationship should be formulated in a co-rotational configuration:

$$\dot{\bar{\tau}}=\bar{\mathrm{C}}^{\tau}:\bar{\mathrm{D}}\text{ or }\dot{\bar{\tau}}_{ij}=\bar{\mathrm{C}}^{\tau}_{\mathrm{ijkl}}:\bar{\mathrm{D}}_{\mathrm{kl}} \quad (11)$$

where $\bar{\tau}=\mathrm{R}^{\mathrm{T}}\,\tau\,\mathrm{R}$ is the co-rotational Kirchhoff stress, $\bar{\mathrm{D}}=\mathrm{R}^{\mathrm{T}}\,\mathrm{D}\,\mathrm{R}$ is the co-rotational rate of deformation tensor, with R being the orthogonal rotational tensor, obtained from the polar decomposition of the deformation tensor F= R·U (where U is the right stretch tensor, which is symmetric). As co-rotated magnitudes remain invariant to rigid body motion, no restrictions are imposed to $\bar{\mathrm{C}}^{\tau}$.

In reference [3], the relationship between the Lie derivative of Kirchhoff stress tensor and the co-rotational Kirchhoff stress was addressed. Furthermore and for the non symmetry arising in the constitutive equations is too strong a disadvantage, an alternative that avoids this situation was addressed through the following expression:

$$\hat{C}^{\tau}_{ijkl}=R_{im}R_{jn}R_{kp}R_{lq}\bar{C}^{\tau}_{mnpq}-\hat{C}^{sim}_{ijkl} \quad (12)$$

The previous is used in Eq. (10) instead of C^{τ}, leading to a symmetric stiffness matrix.

3.3. The Non Lineal Stress Problem in Non Saturated Soils

The general soil stress problem is commonly tackled through an additive decomposition of the efforts on those over the solid phase and over the fluid phase. Bearing in mind the specific case of partially saturated soil, the problem may be carried out in two different ways: Using the additive decomposition of either the Jaumann stress rate of the Kirchhoff tensor or the Lie derivative of Kirchhoff tensor [3]. For the second alternative is a little more straightforward, the attention will be focused in this one. At the outset and relying on Eq. (1), the additive decomposition of the Kirchhoff stress tensor is considered:

$$\tau=\tau'+a_1\tau^w+a_2\tau^g=\tau'+Ja_1p^wI+Ja_2p^gI \quad (13)$$

Substituting Eq. (13) on the expression for the lie derivative of the Kirchhoff stress tensor $L_v\tau=\dot{\tau}-\mathrm{L}\tau-\tau\mathrm{L}$, and performing the time derivative, leads to:

$$L_v\tau = \mathrm{C}^{\tau}\mathrm{D} + J\mathrm{m}(\mathrm{m}^{\mathrm{T}}\mathrm{D})a_1p^w + J\mathrm{m}a_1\dot{p}^w + J\mathrm{m}(\mathrm{m}^{\mathrm{T}}\mathrm{D})a_2p^g + J\mathrm{m}a_2\dot{p}^g \tag{14}$$

where m = {1, 1, 1, 0, 0, 0}. In the above equation, the constitutive Eq. (9) is strictly applied to the solid phase (effective stress). In place of the constitutive tensor C^{τ}, Eq. (12) may be used: For Biot constant, the subsequent expression must be regarded.

$$\alpha = 1 - \mathrm{m}^{\mathrm{T}}\hat{\mathrm{C}}^{\tau}\mathrm{m}/9\,k_s \tag{15}$$

3.4. The Weak Form of the Equilibrium Equation for Non Saturated Soil

An equivalent form of Eq. (8) for non saturated soils may be carried out regarding the stress decomposition yielded in the precedent paragraph.

$$\int_{\varphi(B)} \delta\mathrm{L}: (\mathrm{L}\,\tau + \mathrm{C}^{\tau}: \mathrm{D} + J\,\mathrm{I}(\mathrm{I}: \mathrm{D})a_1p^w + J\,\mathrm{I}a_1\dot{p}^w + J\,\mathrm{I}(\mathrm{I}: \mathrm{D})a_2p^g + J\,\mathrm{I}a_2\dot{p}^g)\, d\varphi(B)/J = \int_{\varphi(B)} \delta\mathrm{v}\,\dot{\mathrm{b}}\rho\, d\varphi(B) + \int_{\partial\varphi(B)} \delta\,\mathrm{v}\,\dot{\bar{\mathrm{t}}}\, d\partial\varphi(B) \tag{16}$$

For the geometric non linearity to be included in the non saturated consolidation phenomena, the above equation should be used in place of first of the system (7).

4. Finite Element Discretization

The weak form of equations (7) may be derived using the general Galerkin method. After applying the finite element method, the following discrete system at element level is presented:

$$\begin{aligned} \mathrm{K}\,\dot{\hat{\mathrm{u}}} + \mathrm{C}_{sw}\dot{\hat{\mathrm{p}}}^w + \mathrm{C}_{sg}\dot{\hat{\mathrm{p}}}^g &= \dot{\mathrm{F}}_s \\ \mathrm{C}_{ws}\,\dot{\hat{\mathrm{u}}} + \mathrm{P}_{ww}\,\dot{\hat{\mathrm{p}}}^w + \mathrm{Q}_{wg}\dot{\hat{\mathrm{p}}}^g + \mathrm{H}_{ww}\hat{\mathrm{p}}^w &= \dot{\mathrm{F}}_w \\ \mathrm{C}_{gs}\,\dot{\hat{\mathrm{u}}} + \mathrm{Q}_{gw}\,\dot{\hat{\mathrm{p}}}^w + \mathrm{P}_{gg}\dot{\hat{\mathrm{p}}}^g + \mathrm{H}_{gg}\hat{\mathrm{p}}^g &= \dot{\mathrm{F}}_g \end{aligned} \tag{17}$$

being $\dot{\hat{\mathrm{u}}}$, $\dot{\hat{\mathrm{p}}}^w$ and $\dot{\hat{\mathrm{p}}}^g$ the nodal velocity, water and air rate pressure vectors, respectively.

All the matrices and vectors derived in the previous may be found in Di Rado [25].

5. Numerical Examples

This example was added in order to show the ability of the model developed in the previous sections to solve geometrically non linear problem in non-saturate soils. The example consists on a strip footing under uniform load.

The problem data are: Wide *10m*, Depth *5m*, Young's modulus *E= 1000 kPa*, Poisson's ratio *ν= 0.4*, Permeability *k = 8.64 E10−5 m/day*, Void ratio *e= 0.9* and Load *Q = 10 kN/m*.

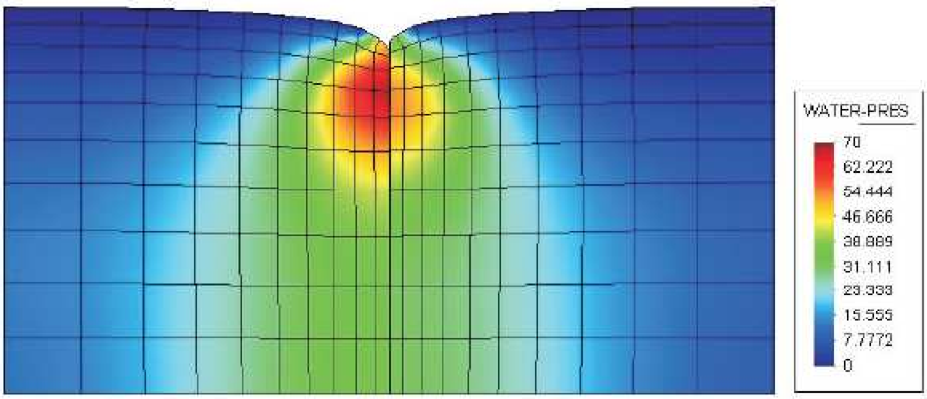

Figure 1. *Surface settlements and water pore pressure for the 20th day.*

In Fig. 1 it can be observed the deformation of the soil skeleton and water pore pressure corresponding to 20 day after the load application, considering small (left side) and large strain situation (right side).

6. Conclusions

A general formulation and the numerical solution for non saturated soils consolidation were presented. The whole system was incorporated into a Galerkin finite element model. The governing equation, in terms of displacement and fluid pressures, result in coupled nonlinear partial differential equation. The use of either complex or simple symmetric formulations with accurate prediction in deformation and pore-pressures is strongly dependent on the soil characteristic curves, their derivatives and the shear stress order. The approach presented here, make possible to easily switch between both kinds of formulations according to the data.

References

[1] Z.Y. Ai, Y.C. Cheng, W.Z. Zeng, C. Wu, 3-D consolidation of multilayered porous medium with anisotropic permeability and compressible pore fluid, *Meccanica* 48 (2013), 491-499

[2] M.A. Biot, General theory of three dimensional consolidation, *J. Appl. Phys.* 12 (1941),155-164

[3] H.A. Di Rado, J.L. Mroginski, P.A. Beneyto, A.M. Awruch, A symmetric constitutive matrix for the nonlinear analysis of hypoelastic solids based on a formulation leading to a non-symmetric stiffness matrix, *Communications in Numerical Methods in Engineering*. 24 (2008), 1079–1092

[4] H. Di Rado, P. Beneyto, J. Mroginski, A. Awruch, Influence of the saturation-suction relationship in the formulation of non-saturated soil consolidation models, *Math. Comput. Model*. 49 (2009) 1058-1070.

[5] J. Ghaboussi, K. Kim, Quasistatic and Dynamic Analysis of Saturated and Partially Saturated Soils. *Mechanics of Engineering Materials*, 14 (1984) 277-296

[6] S. Hassanizadeh, W. Gray, General conservation equation for multiphase systems: 1 Averaging procedure. *Advances in water resources* 2 (1979), 131-144

[7] S. Hassanizadeh, W. Gray, General conservation equation for multiphase systems: 2 Mass, Momenta, Energy and Entropy equations, *Advances in water resources,* 2 (1979), 191-203

[8] S. Hassanizadeh, W. Gray, General conservation equation for multiphase systems: 3 Constitutive theory for porous media flow, *Advances in water resources*, 3 (1980), 25-40

[9] N. Khalili, M.H. Khabbaz, M.H., On the theory of three-dimensional consolidation in unsaturated soils, *First International Conference on Unsaturated Soils - UNSAT'95*, 1995, 745-750.

[10] G. Klubertanz, F. Bouchelaghem, L. Laloui, L. Vulliet, Miscible and Immiscible Multiphase Flow in Deformable Porous Media, *Mathematical and Computer Modeling*. 37 pp. 571-582 (2003).

[11] R.W. Lewis, B.A. Schrefler, *The Finite Element Method in the Deformation and Consolidation of Porous Media*, J. Wiley & Sons, New York. (1987).

[12] R.W. Lewis, B.A. Schrefler, N.A. Rahman, A Finite Element Analysis of Multiphase Immiscible Flow in Deforming Porous Media for Subsurface Systems, *Communications in Numerical Methods in Engineering*, 14 (1998), 135-149.

[13] L.E. Malvern, *Introduction to the Mechanics of a Continuum Medium*. Prentice Hall, USA.(1969).

[14] I. Masters, W.K.S. Pao, and R. Lewis, Coupling temperature to a double porosity model of deformable porous media, *Int. J. Numer. Anal. Meth. Geomech.* 49 (2000), 421-438

[15] R.J. Moitsheki, P. Broadbridge, M.P. Edwards, Symmetry solutions for transient solute transport in unsaturated soils with realistic water profile. *Transport in Porous Media* 61 (2005), 109-125

[16] J.L. Mroginski, H.A. Di Rado, P.A. Beneyto, A.M. Awruch "A finite element approach for multiphase fluid flow in porous media, *Mathematics and Computers in Simulation*. 81 (2010), 76-91

[17] J.L. Mroginski and G. Etse, A finite element formulation of gradient-based plasticity for porous media with C1 interpolation of state variables, *Computers and Geotechnics*, 49 (2013), 7-17

[18] A.K.L. Ng, J.C. Small, Use of coupled finite element analysis in unsaturated soil problems, *Int. J. Numer. Anal. Meth. Geomech*. 24 (2000), 73-94

[19] S. Pietraszezak, G.N. Pandle, On the mechanics of partially saturated soils, *Computers and Geotechnics* 12 (1991), 55-71 (1991)

[20] P. Royer, C. Boutin, Time Analysis of the Three Characteristic Behaviours of Dual-Porosity Media. I: Fluid Flow and Solute Transport, *Transport in porous media*, 95 (2012), 603-626

[21] B.A. Schrefler, Computer modeling in environmental geomechanics, *Computers and structures*, 79 (2001), 2209-2223

[22] J.C. Simo, T.J.R. Hughes, *Computational Inelasticity*. Springer – Verlag. New York, Inc. (1998)

[23] F. Su, F. Larsson, K. Runesson, Computational homogenization of coupled consolidation problems in micro heterogeneous porous media, *Int. J. Num. M. Eng.* 88 (2011), 1198–1218

[24] P. Tritscher, W.W. Read, P. Broadbridge, J.H. Knight, Steady saturated-unsaturated flow in irregular porous domains. *Mathematical and Computer Modelling* 34 (2001), 177-194

[25] H. A. Di Rado, "Numerical Simulation of physical and geometric nonlinearities. None saturated soil consolidation analysis" (Spanish). *Doctoral Thesis, Northeast National University. Argentine*, Ch. 5 (2006),121-133