# Linear fractional multi-objective optimization problems subject to fuzzy relational equations with the max-average composition

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**Abstract:** In this paper, linear fractional multi-objective optimization problems subject to a system of fuzzy relational equations (FRE) using the max-average composition are considered. First, some theorems and results are presented to thoroughly identify and reduce the feasible set of the fuzzy relation equations. Next, the linear fractional multi-objective optimization problem is converted to a linear one using Nykowski and Zolkiewski's approach. Then, the efficient solutions are obtained by applying the improved $\varepsilon$-constraint method. Finally, the proposed method is effectively tested by solving a consistent test problem.

**Keywords:** Fuzzy Relational Equation, The Max-Average Composition,
Linear Fractional Multi-Objective Optimization Problems, The Improved $\varepsilon$-Constraint Method

## 1. Introduction

Multi-objective programming problems play an important role in the optimization theory. Generally speaking, the objective functions of a multiple objective programming (MOP) problem may conflict with one another. Thus, the notion of Pareto optimality or efficiency associated with the feasible region has been introduced. There are several methods to find the efficient solutions of the MOP in the literature. For details see [6].

An optimization problem with multiple objective functions and fuzzy relational equation (FRE) or fuzzy relational inequality (FRI) constraints is among interesting issues in this field. Wang [23] considered a multi-objective mathematical programming problem with constraints defined by the FRE with the max-min composition. The nonlinear multi-objective optimization problems subject to FRE with the max-min and max-average compositions were studied in [16] and [13,14], respectively. They developed specific reduction procedures to simplify a given problem, according to the special structure of the solution set, and further proposed some genetic algorithms to attain efficient

solutions. Many researchers have considered the problem and developed the theoretical topics, solution procedures and various applications [1-5,8-12,17,18,20,22,24-26].

Here, a linear fractional multi-objective optimization problem subject to the FRE with the max-average composition is considered. Since the feasible domain of this problem is generally non-convex, traditional methods may have difficulty in deriving the set of efficient solutions. Nevertheless, here, an efficient method is proposed to obtain the efficient solutions of the linear fractional multi-objective optimization problem (LFMOP) subject to the FRE, exactly and completely, using special structures of the feasible domain and the objective functions. Indeed, the linear fractional multi-objective optimization problem is converted to a linear one using Nykowski and Zolkiewski's approach [19]. Then, the efficient solutions are obtained by applying the improved $\varepsilon$-constraint method.

In Section 2, the basic definitions and theorems are presented. In Section 3, the feasible set of systems of fuzzy relation equations will be characterized and the corresponding reduced problem will be investigated (our motivation). Section 4 will present our proposed approach to solve a linear fractional multi-objective optimization problem

with fuzzy relation equation constraints. To illustrate the procedure, a numerical example will be provided in Section 5 that will be followed by our concluding remarks in Section 6.

# 2. Preliminaries

In this section, some definitions and theorems are presented which are basic in the literature [6]. Consider,

$$\min_{x \in \chi} z = f(x) = \left(f_1(x), \dots, f_p(x)\right) \qquad (2.1)$$

where $f: \mathcal{R}^n \longrightarrow \mathcal{R}^p$ is a nonlinear vector function and $\chi \subseteq \mathcal{R}^n$ is the set of feasible solutions. Moreover, $x \in \chi$ and $Z = \left\{z \in \mathcal{R}^p : \exists x \in \chi, z = \left(f_1(x), \dots, f_p(x)\right)\right\}$ are called a solution vector and the criterion space, respectively.

*Notation 2.1.* Let $x, y \in \mathcal{R}^n$,

$$x \lneqq y \iff x_j \leq y_j, \text{ for } j = 1, \dots, n.$$

$$x \leq y \iff x \lneqq y \text{ and } x \neq y.$$

$$x < y \iff x_j < y_j, \text{ for } j = 1, \dots, n.$$

Let $z^1, z^2 \in Z$, we say $z^1$ dominates $z^2$ and $z^1$ is non-dominated if and only if $z^1 \leq z^2$ and there is no $z \in Z$ that dominates $z^1$, respectively. Moreover, we say $\tilde{x} \in \chi$ is an efficient solution to problem (2.1) if and only if there is no $x \in \chi$ such that $f(x) \leq f(\tilde{x})$.

In the remainder of the present section, some definitions, notations and theorems from different topics are reviewed which are needed in the latter sections. Let $I = \{1, \dots, m\}, J = \{1, \dots, n\}, P = \{1, \dots, p\}$, $A = \left(a_{ij}\right)_{m \times n}$ and $b = (b_i)_{m \times 1}, a_{ij}, b_i \in [0,1]$, for all $i \in I$ and $j \in J$. A system of fuzzy relation equations with the max-average composition can be considered as

$$A o_{av} x = b, \qquad (2.2)$$

where the operator "$o_{av}$" is defined as follows:

$$a_i o_{av} x = \max_{j \in J} \frac{a_{ij} + x_j}{2}, \quad i \in I, \qquad (2.3)$$

and $a_i$ is the $i$-th row of $A$.

*Notation 2.2.* For the $i$-th constraint (2.2), we set

$$J_i^1 = \left\{j \in J: a_{ij} < 2b_i\right\},$$

$$J_i^2 = \left\{j \in J: a_{ij} = 2b_i\right\},$$

$$J_i^3 = \left\{j \in J: a_{ij} > 2b_i\right\},$$

$$G(i) = \left\{j \in J: 2b_i - a_{ij} \leq 1\right\}, \qquad i \in I$$

$$\mathcal{I}(j) = \{i \in I: j \in G(i)\}, \qquad j \in J$$

$$\chi^i = \{x \in [0,1]^n: a_i o_{av} x = b_i\}, \qquad i \in I$$

$$\chi = \{x \in [0,1]^n: A o_{av} x = b\}.$$

$\chi$ and $\chi^i$ are the feasible sets of the problem and the $i$-th constraint, respectively. When $\chi$ is not empty, it is in general a non-convex and non-singleton set and can be completely determined by a unique maximum and a finite number of minimal solutions [21].

*Definition 2.3.* [15]. $\hat{x} \in \chi$ is a maximum solution if $x \leq \hat{x}$, for all $x \in \chi$, and if $x \leq \check{x}$ implies $x = \check{x}$, for all $x \in \chi$, $\check{x} \in \chi$ is called a minimal solution.

*Definition 2.4.* [16]. If a value-change in some element(s) of a given fuzzy relation matrix $A$ has no effect on the solutions of a corresponding fuzzy relation equations, this value-change is called an *equivalence operation*.

A traditional approach for solving the MOP is the scalarization techniques that formulate the MOP as a single objective program. Sometimes, the feasible set, $\chi$, is limited by some new constraints related to objective functions of the MOP and/or some new variables introduced. Here, we utilize a well-known scalarization technique called the $\varepsilon$-constraint method which was improved by Ehrgott and Ruzika [7]. In this method, the corresponding single program to the MOP (2.1) is,

$$\min f_k(x) - \sum_{i \neq k} \lambda_i s_i^+ + \sum_{i \neq k} \mu_i s_i^-$$
$$s.t.$$
$$f_i(x) + s_i^+ - s_i^- \leq \varepsilon_i, \quad i \in P \setminus \{k\} \qquad (2.4)$$
$$s_i^+, s_i^- \geq 0, \qquad i \in P \setminus \{k\}$$
$$x \in \chi,$$

where $\lambda_i, \mu_i \geq 0$, for all $i \in P \setminus \{k\}$.

The following results on the improved $\varepsilon$-constraint method can be found in [7]:

(i) If there is an $i \neq k$ such that $\mu_i - \lambda_i < 0$, then problem (2.4) will be unbounded.

(ii) If $\mu - \lambda \geq 0$, then there is always an optimal solution of (2.4) such that $s_i^+ s_i^- = 0, i \neq k$.

(iii) Let $(\lambda, \mu) \geqq 0$. If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an optimal solution of (2.4), then $\hat{x}$ will be a weakly efficient solution of the MOP.

(iv) Let $(\lambda, \mu) \geqq 0$. Let $(\hat{x}, \hat{s}^+, \hat{s}^-)$ be an optimal solution of (2.4). If $\hat{x}$ is unique, then $\hat{x}$ will be a strictly efficient solution of the MOP.

(v) Let $(\lambda, \mu) > 0$. Let $(\hat{x}, \hat{s}^+, \hat{s}^-)$ be an optimal solution of (2.4). Then, $\hat{x}$ is an efficient solution of the MOP.

We shall assume throughout the paper that $\mu - \lambda \geq 0$, and $m$, $n$ and $p$ stand for the number of constraints in system (2.2), the dimension of solution vectors and the number of objective functions of problem (2.1), respectively.

# 3. Systems of Fuzzy Relation Equations

Here, the feasible set of systems of fuzzy relation equations is characterized and the corresponding reduced problem will be investigated. It means we consider the characteristics of the solution set of (2.2), when $x \in [0,1]^n$, that is,

$$A o_{av} x = b, \\ x \in [0,1]^n, \qquad (3.1)$$

and attempt to simplify the problem by reducing the solution set. System (3.1) can also be considered as $a_i o_{av} x = b_i$, $i \in I$, where $a_i$ is the $i$-th row of A where the operator "$o_{av}$"

is defined by (2.3). The results which are proven in this section are our contributions.

### 3.1. Characterization of the Feasible Set

*Proposition 3.1.* A vector $x \in [0,1]^n$ fulfills the $i$-th constraint if and only if

$$\forall j \in J, \ x_j \leq 2b_i - a_{ij},$$
$$\exists j_0 \in J, \ x_{j_0} = 2b_i - a_{ij_0}. \tag{3.2}$$

Lemma 3.2. If $J_i^3 \neq \emptyset$ then $\chi^i = \emptyset$.

Proof. Let $j_0 \in J_i^3$ . By contraction, if $x \in \chi^i$ then according to (2.3), $x_j \leq 2b_i - a_{ij}$ , for all $j \in J$ . Thus, $x_{j_0} \leq 2b_i - a_{ij_0} < 0$.

Corollary 3.3. If $\chi^i \neq \emptyset$ then $J_i^3 = \emptyset$.

Lemma 3.4. [12]. Let $\chi^i \neq \emptyset$. The maximum solution in $\chi^i$ is $\ ^i\hat{x} = (\ ^i\hat{x}_1, \dots, \ ^i\hat{x}_n)$ where $\ ^i\hat{x}_j = \min(2b_i - a_{ij}, 1)$, for all $j \in J$.

Proof. $\ ^i\hat{x} \in \chi$ was shown in [12]. By contraction, we prove that $\ ^i\hat{x}$ is the maximum solution of $\chi^i$ . Let $x \in \chi^i$ and $j_0 \in J$ such that $x_{j\_0} > \ ^i\hat{x}_{j\_0} = \min(2b\_i - a_{ij_0}, 1)$ . Thus, $x_{j_0} > 2b_i - a_{ij_0}$ or $x_{j\_0} > 1$ (this case is impossible, since $x_{j\_0} \in [0,1]$). If $x_{j_0} > 2b_i - a_{ij_0}$ , we have $\max_{j \in J}(a_{ij} + x_j) > 2b_i$ that is $x \notin \chi^i$.

Notation 3.5. Let $\ ^i\breve{x}(j) = (\ ^i\breve{x}(j)_1, \dots, \ ^i\breve{x}(j)_n)$ where $j \in G(i)$ (see Notation 2.2) and

$$\ ^i\breve{x}(j)_k = \begin{cases} 2b_i - a_{ij}, & k = j, \\ 0, & k \neq j, \end{cases} \tag{3.3}$$

Lemma 3.6. [12]. Let $\chi^i \neq \emptyset$.
a. For all $j \in G(i)$, $\ ^i\breve{x}(j)$ is a minimal solution in $\chi^i$.
b. $\chi^i = \bigcup_{j \in G(i)} [\ ^i\breve{x}, \ ^i\hat{x}]$.
Definition 3.7.

$$\hat{x} = \min_{i \in I} \ ^i\hat{x} \tag{3.4}$$

$$\breve{x} = \max_{i \in I} \ ^i\breve{x}(e(i)) \tag{3.5}$$

where $e = (e(1), \dots, e(m))$ and $e(i) \in G(i)$ for all $i \in I$.

Lemma 3.8. [12].
a. $\hat{x}$ is the maximum solution of (3.1).
b. $\chi_0 \subseteq \bar{X}$, where $\chi_0$ denotes the minimal solutions set of $\chi$, $\bar{X} = \{\breve{x}(e) : e \in E\}$ and $E = G(1) \times \dots \times G(m)$.
c. $\chi = \bigcup_{e \in E} [\breve{x}(e), \hat{x}]$, where $E = \{e = (e(1), \dots, e(m)) : e(i) \in G(i), i = 1, \dots, m\}$.

Lemma 3.9. If there is an $i \in I$ such that $J_i^3 \neq \emptyset$ then $\chi = \emptyset$.

Proof. According to Lemma 3.2, the proof is obvious.

Theorem 3.10. $\chi = \bigcap_{i \in I} \chi^i$ .

*Proof.* Let $E = \{e = (e(1), \dots, e(m)) : e(i) \in G(i), i = 1, \dots, m\}$  It is sufficient to prove two statements: (1)$\chi \subseteq \bigcap_{i \in I} \chi^i$ and (2)$\bigcap_{i \in I} \chi^i \subseteq \chi$.

*For (1):*

$$x \in \chi \iff x \in \bigcup_{e \in E} [\breve{x}(e), \hat{x}]$$

$$\iff \exists e \in E, \breve{x}(e) \leq x \leq \hat{x}$$

$$\iff \exists e \in E, \ \forall k \in J,$$

$$(\breve{x}(e))_k = \max_{i \in I}(\ ^i\breve{x}(e(i)))_k \leq x_k \leq \hat{x}_k = \min_{i \in I} \ ^i\hat{x}_k$$

$$\iff \exists e \in E, \ \forall k \in J, \forall i \in I,$$

$$\left(\ ^i\breve{x}(e(i))\right)_k \leq x_k \leq \ ^i\hat{x}_k$$

$$\iff \exists e \in E, \forall k \in J, \forall i \in I,$$

$$\begin{cases} 2b_i - a_{ik}, & k = e(i) \\ 0, & \text{otherwise} \end{cases} \leq x_k \leq \min(2b_i - a_{ik}, 1)$$

$$\tag{3.6}$$

$$\iff \exists e \in E, \forall k \in J, \forall i \in I,$$

$$\begin{cases} 2b_i - a_{ik}, & k = e(i) \\ 0, & \text{otherwise} \end{cases} \leq x_k \leq \begin{cases} 2b_i - a_{ik}, & k \in G(i) \\ 1, & \text{otherwise} \end{cases}$$

$$\iff \forall q \in I, \exists e(q) \in G(q), \forall k \in J, \forall i \in I,$$

$$\begin{cases} 2b_i - a_{ik}, & k = e(i) \\ 0, & \text{otherwise} \end{cases} \leq x_k \leq \begin{cases} 2b_i - a_{ik}, & k \in G(i) \\ 1, & \text{otherwise} \end{cases}$$

$$\implies \forall q \in I, \exists e(q) \in G(q), \text{Let } j = e(q), \forall k \in J, \forall i = q,$$

$$\begin{cases} 2b_i - a_{ik}, & k = j \\ 0, & \text{otherwise} \end{cases} \leq x_k \leq \begin{cases} 2b_i - a_{ik} & 2b_i - a_{ik} \leq 1 \\ 1, & 2b_i - a_{ik} > 1 \end{cases}$$

$$\iff \forall i \in I, \exists j \in G(i), \forall k \in J,$$

$$\left(\ ^i\breve{x}(j)\right)_k \leq x_k \leq \min(2b_i - a_{ik}, 1) = \ ^i\hat{x}_k$$

$$\tag{3.7}$$

$$\iff \forall i \in I, \exists j \in G(i), x \in [\ ^i\breve{x}(j), \ ^i\hat{x}]$$

$$\iff \forall i \in I, x \in \bigcup_{j \in G(i)} [\ ^i\breve{x}(j), \ ^i\hat{x}]$$

$$\iff \forall i \in I, x \in \chi^i$$

$$\iff x \in \bigcap_{i \in I} \chi^i$$

*For (2):* Like to the previous part, we have:

$$x \in \bigcap_{i \in I} \chi^i \iff$$

$$\dots \quad \iff (3.7) \quad \iff \quad \forall i \in I, \exists j_i \in G(i), \forall k \in J,$$

$$\begin{cases} 2b_i - a_{ik}, & k = j_i \\ 0, & \text{otherwise} \end{cases} \leq x_k \leq \begin{cases} 2b_i - a_{ik}, & k \in G(i) \\ 1, & \text{otherwise} \end{cases}$$

$$\iff \forall i \in I, \exists j_i = e(i) \in G(i), \forall k \in J,$$

$$\begin{cases} 2b_i - a_{ik}, & k = e(i) \\ 0, & \text{otherwise} \end{cases} \leq x_k \leq \begin{cases} 2b_i - a_{ik}, & k \in G(i) \\ 1, & \text{otherwise} \end{cases}$$

$$\implies \text{Let } e = (e(1), \dots, e(m)), \forall i \in I, \forall k \in J,$$

$$\begin{cases} 2b_i - a_{ik}, & k = e(i) \\ 0, & \text{otherwise} \end{cases} \leq x_k \leq \begin{cases} 2b_i - a_{ik}, & k \in G(i) \\ 1, & \text{otherwise} \end{cases}$$

$$\Leftrightarrow \quad (3.6) \Leftrightarrow \cdots \Leftrightarrow x \in \chi$$

*Remark 3.11.* If $b_i = 0$ for some $i \in I$, it can be shown that $\chi = \emptyset$.

According to Remark 3.11 and Lemma 3.9, we shall assume in this section that $b > 0$ and $J_i^3 = \emptyset$, for all $i \in I$, respectively.

### 3.2. Problem Reduction

*Detection of redundant constraints:* Here, we consider the conditions under which a constraint of the problem can be omitted.

Theorem 3.12. Let $\chi \neq \emptyset$, $i_1, i_2 \in I$ and $i_1 \neq i_2$. If the $i_1$-th and $i_2$-th constraints satisfy

1. $G(i_1) = G(i_2)$,
2. $\forall k \in G(i_1) = G(i_2)$, $2b_{i_1} - a_{i_1 k} = 2b_{i_2} - a_{i_2 k}$,

then the constraint $i_2$ (or $i_1$) is irredundant.

*Proof.* We show that $\bigcap_{i \neq i_2} \chi^i \subseteq \chi^{i_2}$. Let $x \in \bigcap_{i \neq i_2} \chi^i$ be arbitrary.

$$x \in \bigcap_{i \neq i_2} \chi^i$$

$$\Leftrightarrow \quad \forall i \in I \setminus \{i_2\}, x \in \chi^i = \bigcup_{j \in G(i)} [\ ^i \check{x}(j), \ ^i \hat{x}]$$

$$\Leftrightarrow \quad \forall i \in I \setminus \{i_2\}, \exists j_i \in G(i), \ ^i \check{x}(j_i) \leq x \leq \ ^i \hat{x}$$

$$\Leftrightarrow \quad \forall i \in I \setminus \{i_2\}, \exists j_i \in G(i), \forall k \in J, \left( \ ^i \check{x}(j_i) \right)_k \leq x_k \leq \ ^i \hat{x}_k$$

$$\Leftrightarrow \quad \forall i \in I \setminus \{i_2\}, \exists j_i \in G(i), \forall k \in J,$$
$$\begin{cases} 2b_i - a_{ij_i}, & k = j_i \\ 0, & \text{otherwise} \end{cases} \leq x_k \leq \ ^i \hat{x}_k = \min(2b_i - a_{ik}, 1)$$

$$\Leftrightarrow \quad \forall i \in I \setminus \{i_2\}, \exists j_i \in G(i), \forall k \in J,$$
$$\begin{cases} 0 \leq x_k \leq 1, & k \notin G(i), \\ 0 \leq x_k \leq 2b_i - a_{ik}, & k \in G(i) \setminus \{j_i\}, \\ 2b_i - a_{ij_i} \leq x_k \leq 2b_i - a_{ik}, & k = j_i \end{cases}$$

$$\Leftrightarrow \quad \forall i \in I \setminus \{i_2\}, \exists j_i \in G(i),$$
$$\begin{cases} x_k \leq 2b_i - a_{ik}, & k \in G(i) \setminus \{j_i\}, \\ x_k = 2b_i - a_{ij_i}, & k = j_i, \end{cases}$$

$$\Rightarrow \quad i = i_1, \exists j_{i_1} \in G(i_1),$$
$$\begin{cases} x_k \leq 2b_{i_1} - a_{i_1 k}, & k \in G(i_1) \setminus \{j_{i_1}\}, \\ x_{j_{i_1}} = 2b_{i_1} - a_{i_1 j_{i_1}}, & k = j_{i_1}, \end{cases}$$

$$\Leftrightarrow \quad \exists j_{i_1} \in G(i_2), \begin{cases} x_k \leq 2b_{i_2} - a_{i_2 k}, & k \in G(i_2) \setminus \{j_{i_1}\}, \\ x_{j_{i_1}} = 2b_{i_2} - a_{i_2 j_{i_1}}, & k = j_{i_1}, \end{cases}$$

$$\Leftrightarrow \quad \exists j_{i_1} \in G(i_2), \forall k \in J,$$
$$\begin{cases} 0 \leq x_k \leq 1, & k \notin G(i_2), \\ 0 \leq x_k \leq 2b_{i_2} - a_{i_2 k}, & k \in G(i_2) \setminus \{j_{i_1}\}, \\ x_{j_{i_1}} = 2b_{i_2} - a_{i_2 j_{i_1}}, & k = j_{i_1}, \end{cases}$$

$$\Leftrightarrow \quad \exists j_{i_1} \in G(i_2), \forall k \in J, \left( \ ^{i_2} \check{x}(j_{i_1}) \right)_k \leq x_k \leq \ ^{i_2} \hat{x}_k$$

$$\Leftrightarrow \quad x \in \bigcup_{j \in G(i_2)} [\ ^{i_2} \check{x}(j), \ ^{i_2} \hat{x}] = \chi^{i_2}$$

*Example 3.13.* Consider,

$$\begin{pmatrix} 0.7954 & 0.4927 & 0.3547 \\ 0.7751 & 0.2369 & 0.8448 \\ 0.8165 & 0.5139 & 0.3759 \end{pmatrix} o_{av} x = \begin{pmatrix} 0.6391 \\ 0.6289 \\ 0.6496 \end{pmatrix}.$$

Here, we have $G(1) = G(3) = \{1,2,3\}$. Computing $w_{ij} = 2b_i - a_{ij}$, for all $i \in I$ and $j \in J$ results that:

$$W = \begin{pmatrix} 0.4827 & 0.7854 & 0.9234 \\ 0.4827 & 1.0209 & 0.4130 \\ 0.4827 & 0.7854 & 0.9234 \end{pmatrix}.$$

These values of the first and third rows are equal, thus, by applying Theorem 3.12, the third (or the first) constraint is omitted. By solving the reduced system, we obtain the maximum solution, $\hat{x} = (0.4827, 0.7854, 0.4130)^t$ and the minimal solutions as follows:

$$\check{x}((1,1,1)) = (0.4827, 0, 0)^t,$$

$$\check{x}((1,1,2)) = (0.4827, 0.7854, 0)^t,$$

$$\check{x}((1,3,1)) = (0.4827, 0, 0.4130)^t.$$

*Detection of fixed components:* By the following lemmas, the $j$-th element of solutions can be fixed and be eliminated from the solution space.

*Lemma 3.14.* [12]. If for some $j \in J$, $\ ^i \check{x}(j) = \bar{0}$ or $j \in J_i^2$ then the equation $i$ has just one minimum as $\ ^i \check{x} = \bar{0}$.

*Lemma 3.15.* For a constraint, say $i$, if $J_i^2 \neq \emptyset$, then $x_j = 0$ for all $x \in \chi$ and all $j \in J_i^2$. It means the component(s) $j$, $j \in J_i^2$, can be eliminated from the solution space.

*Proof.* Let $x \in \chi = \bigcap_{k \in I} \chi^k$ be arbitrary.

$$x \in \bigcap_{k \in I} \chi^k \quad \Rightarrow \quad x \in \chi^i$$

$$\Leftrightarrow \quad \begin{cases} \forall j \in J, & x_j \leq 2b_i - a_{ij}, \\ \exists j \in J, & x_j = 2b_i - a_{ij}, \end{cases}$$

$$\Rightarrow \quad \forall j \in J_i^2, x_j \leq 2b_i - a_{ij} = 0,$$
$$\Rightarrow \quad \forall j \in J_i^2, x_j = 0.$$

*Example 3.16.* Consider the following feasible system

$$\begin{pmatrix} 0.2876 & 0.0912 & 0.5763 & 0.6834 \\ 0.5466 & 0.4258 & 0.6445 & 0.6477 \\ 0.1917 & 0.1485 & 0.4579 & 0.7245 \\ 0.7093 & 0.2363 & 0.1195 & 0.6073 \end{pmatrix} o_{av} x = \begin{pmatrix} 0.4214 \\ 0.4851 \\ 0.3622 \\ 0.5664 \end{pmatrix}$$

Since $J_3^2 = \{4\}$, according to Lemma 3.15, the fourth component of any solution vector is fixed to zero. Therefore, this component can be eliminated from the solution space as well as the fourth column of the coefficient matrix from consideration. The maximum and minimal solutions are $\hat{x} = (0.4235, 0.5444, 0.2665, 0)^t$ and $\check{x} =$

$(0.4236, 0, 0.2666, 0)^t$ , respectively.

*Lemma 3.17.* Let $\chi \neq \emptyset$. If there are $i \in I$ and $j \in J$ such that $b_i = a_{ij} = 1$, then the $j$-th element of solutions can be fixed as one and be eliminated from the solution space.

*Proof.* We have:

$$\chi^i = \{x \in [0,1]^n : \max_{k \in J}(a_{ik} + x_k) = 2b_i$$
$$= \{x \in [0,1]^n : \forall k \in J, a_{ik} + x_k \leq 2, \exists j' \in J, a_{ij'} + x_{j'} = 2\}.$$

Since $a_{ik}, x_k \in [0,1]$, for all $k \in J$ and $a_{ij} = 1$, the set $\chi^i$ can be simplified as follows:

$$\chi^i = \{x \in [0,1]^n : (\text{let } j' = j)a_{ij} + x_j = 2\}$$
$$= \{x \in [0,1]^n : x_j = 2 - a_{ij} = 1\}$$

Example 3.18. Consider,

$$\begin{pmatrix} 0.4425 & 1.0000 & 0.3593 \\ 0.7364 & 0.3948 & 0.6834 \\ 0.7041 & 0.4424 & 0.0197 \end{pmatrix} 0_{av} x = \begin{pmatrix} 1.0000 \\ 0.8346 \\ 0.8185 \end{pmatrix}$$

By Lemma 3.17, the second element of solutions can be fixed as one and be eliminated from the solution space, since $a_{12} = b_1 = 1.0000$. The maximum and minimal solutions are                $\hat{x} = (0.9329, 1.0000, 0.9858)^t$                and $\check{x} = (0.9329, 1.0000, 0)^t$, respectively.

*Lemma 3.19.* Suppose $G(i) = \{j_i\}$ (a singleton set), for all $i \in I$. If there is a $j_0 \in J$ such that $\mathcal{I}(j_0) \neq \emptyset$ (see Notation 2.2), then

(1) $2b_{i_0} - a_{i_0 j_0} = 2b_{i_1} - a_{i_1 j_0}$, for all $i_0, i_1 \in \mathcal{I}(j_0)$.

(2) for all $x \in \chi$, the $j_0$-th element of solutions can be fixed as $2b_i - a_{ij_0}, i \in \mathcal{I}(j_0)$, and the component $j_0$ can be eliminated from the solution space.

Proof. (1) If $\mathcal{I}(j_0)$ is singleton then $i_0 = i_1$ and the proof is trivial. Now, assume $i_0, i_1 \in \mathcal{I}(j_0)$ and $i_0 \neq i_1$ . Let $x = (x_1, \dots, x_n) \in \chi$ be an arbitrary feasible solution. We have:

$$\max_{j \in J}(a_{ij} + x_j) = 2b_i \qquad\qquad i = i_0, i_1$$

$$\Leftrightarrow \begin{cases} \forall j \in J, a_{ij} + x_j \leq 2b_i \\ \exists \bar{j}_i \in J, a_{i\bar{j}_i} + x_{\bar{j}_i} = 2b_i \end{cases} \quad i = i_0, i_1$$

$$\Leftrightarrow \begin{cases} \forall j \in J \setminus \{j_i\}, & x_j \leq 1 < 2b_i - a_{ij}(\text{because } j \notin G(i)) \\ j = j_i, & x_j \leq 2b_i - a_{ij} \leq 1 \\ \exists \bar{j}_i \in J, & x_{\bar{j}_i} = 2b_i - a_{i\bar{j}_i} \end{cases}$$
$$i = i_0, i_1.$$

Therefore, it should be $\bar{j}_i = j_i$, for $i = i_0, i_1$. On the other hand, by the assumption, we have $j_{i_0} = j_{i_1} = j_0$ . Thus, $x_{j_{i_0}} = x_{j_{i_1}} = x_{j_0}$, that is, $2b_{i_0} - a_{i_0 j_0} = 2b_{i_1} - a_{i_1 j_0}$.

(2) By assumptions, the vector $e$ is unique, $e = (e(1), \dots, e(m)) = (j_1, \dots, j_m)$ . It is sufficient to prove $\hat{x}_j = (\check{x}(e))_j$:

$\hat{x}_j = (\check{x}(e))_j$

$\Leftrightarrow \min_{i \in I} {}^i\hat{x}_j = \max_{i \in I} \left( {}^i\check{x}(e(i)) \right)_j = \max_{i \in I} \left( {}^i\check{x}(j_i) \right)_j$

$\Leftrightarrow \min_{i \in I}(\min(2b_i - a_{ij}, 1)) = \max_{i \in I} \begin{cases} 2b_i - a_{ij}, & j = j_i, \\ 0, & \text{otherwise,} \end{cases}$

$\Leftrightarrow \min_{i \in \mathcal{I}(j)}(2b_i - a_{ij}) = \max_{i \in \mathcal{I}(j)}(2b_i - a_{ij}).$

According to (1), the last statement is true.

*Example 3.20.* Consider,

$$\begin{pmatrix} 0.1949 & 0.7701 & 0.1171 \\ 0.1759 & 0.0602 & 0.0195 \\ 0.0773 & 0.9243 & 0.2973 \end{pmatrix} o_{av} x = \begin{pmatrix} 0.6242 \\ 0.5635 \\ 0.7013 \end{pmatrix}.$$

By computing $2b_i - a_{ij}$, for all $i, j$, we have $G(1) = \{2\}$, $G(2) = \{1\}$, $G(3) = \{2\}$, $\mathcal{I}(1) = \{2\}$, $\mathcal{I}(2) = \{1,3\}$ and $\mathcal{I}(3) = \{\}$. By Lemma 3.19, the first and second elements of solutions can be fixed as 0.9510 and 0.4782 , respectively, and can be eliminated from the solution space. The maximum and minimal solutions are                $\hat{x} = (0.9510, 0.4782, 1.0000)^t$                and $\check{x} = (0.9510, 0.4782, 0)^t$, respectively.

*Corollary 3.21.* If there is a $j \in J$ such that the values of $2b_i - a_{ij}$ are equal for all $i \in I$, that is:

$$\exists j \in J, \forall i_1, i_2 \in I, 2b_{i_1} - a_{i_1 j} = 2b_{i_2} - a_{i_2 j},$$

then the $j$-th element of solutions can be fixed as $2b_i - a_{ij}$ and be eliminated from the solution space.

According to this corollary, in Example 3.13 the first component of solutions can be fixed as 0.4827 and be eliminated from the solution space. Note that when omitting a column of the coefficient matrix causes a zero row, the corresponding constraint is redundant and should be eliminated from consideration.

*Corollary 3.22.* If $J_i^2 \neq \emptyset$, for all $i \in I$ and $\bigcup_{i \in I} J_i^2 = J$ then $\hat{x} = \check{x} = 0$. It means $\chi = \{\bar{0}\}$.

*Identification of equivalence operation:*

Corollary 3.23. [12]. When for some $i \in I$ and $j \in J$, $j \notin G(i)$, then the value of $a_{ij}$ has no effect on the solution space of (3.1) and can be changed to zero. In other words, changing $a_{ij}$ to zero is an equivalence operation (see Definition 2.4).

Corollary 3.24. [12]. When for some $j \in J$ and $i_1, i_2 \in I$, $2b_{i_1} - a_{i_1 j} < 2b_{i_2} - a_{i_2 j}$, then the value of $a_{i_2 j}$ has no effect on the solution space of (3.1) and can be changed to zero. In other words, changing $a_{i_2 j}$ to zero is an equivalence operation.

*Proof.* It is sufficient to prove $a_{i_2 j} + x_j < 2b_{i_2}$, for all $x \in \chi$. Let $x \in \chi$ be arbitrary. Since $\max_{k \in J}(a_{i_2 k} + x_k) = 2b_{i_2}$, we have $a_{i_2 k} + x_k \leq 2b_{i_2}$, for all $k \in J$, thus, $a_{i_2 j} + x_j \leq 2b_{i_2}$. On the other hand, if $a_{i_2 j} + x_j = 2b_{i_2}$, then according to the assumption we have $a_{i_1 j} + x_j = a_{i_1 j} + 2b_{i\_2} - a_{i_2 j} > 2b_{i_1}$ . Hence, $\max_{k \in J}(a_{i_2 k} + x_k) \geq a_{i_1 j} + x_j > 2b_{i_1}$ , which contradicts with $x \in \chi^{i_1}$ and therefore, $a_{i_2 j} + x_j < 2b_{i\_2}$, for all $x \in \chi$.

*Example 3.25.* Consider the following feasible system

$$\begin{pmatrix} 0.1382 & 0.5078 & 0.8567 \\ 0.3844 & 0.6957 & 0.6279 \\ 0.4504 & 0.4737 & 0.9497 \end{pmatrix} o_{av} x = \begin{pmatrix} 0.7122 \\ 0.5978 \\ 0.7587 \end{pmatrix}. \quad (3.8)$$

Computing $w_{ij} = 2b_i - a_{ij}$, for all $i \in I$ and $j \in J$ results in:

$$W = \begin{pmatrix} 1.2862 & 0.9166 & 0.5677 \\ 0.8113 & 0.4999 & 0.5677 \\ 1.0670 & 1.0438 & 0.5677 \end{pmatrix}.$$

According to Corollary 3.23, changing $a_{11}$, $a_{31}$ and $a_{32}$ to zero is an equivalenceoperation, since $1 \notin G(1) = \{2,3\}$, $1 \notin G(3) = \{3\}\$$ and $2 \notin G(3)$. Additionally, by Corollary 3.24, the value of $a_{12}$ has no effect on the solution space of (3.8) and can be changed to zero (since $W_{12} = 0.9166 > W_{22} = 0.4999$). Thus, the system converts to

$$\begin{pmatrix} 0.0000 & 0.0000 & 0.8567 \\ 0.3844 & 0.6957 & 0.6279 \\ 0.0000 & 0.0000 & 0.9497 \end{pmatrix} o_{av} x = \begin{pmatrix} 0.7122 \\ 0.5978 \\ 0.7587 \end{pmatrix}.$$

By solving the reduced system, we obtain the maximum solution,

$$\hat{x} = (0.8113, 0.4999, 0.5677)^t,$$

and the minimal solutions as follows:

$$\check{x}\big((3,1,3)\big) = (0.8113, 0, 0.5677)^t,$$

$$\check{x}\big((3,2,3)\big) = (0, 0.4999, 0.5677)^t,$$

$$\check{x}\big((3,3,3)\big) = (0, 0, 0.5677)^t.$$

# 4. Linear Fractional Multi-objective Optimization Problems Subject to Fuzzy Relational Equations

Here, we consider

$$\min f(x) = \left( \frac{P_1(x)}{Q_1(x)}, \dots, \frac{P_p(x)}{Q_p(x)} \right)$$

$$\text{s.t.}$$

$$Ao_{av} x = b \quad (4.1)$$

$$x \in [0,1]^n$$

where $P_k(x) = c_k^t x + c_k^0$, $Q_k(x) = d_k^t x + d_k^0$, $k = 1, \dots, p$, $p \geq 2$, $c_k^t = (c_{k1}, \dots, c_{kn}) \in \mathcal{R}^n$, $d_k^t = (d_{k1}, \dots, d_{kn}) \in \mathcal{R}^n$, $c_k^0$ and $d_k^0$ are two constant numbers and $o_{av}$ is the max-average composition. Using the improved $\varepsilon$-constraint method, we try to find the efficient solutions.

First, some assumptions and results are mentioned on the characteristic of (4.1). We denote by $\chi^f$, the set of all efficient solutions of (4.1), that is

$$\chi^f = \{ \bar{x} \in [0,1]^n : \nexists x \in \chi, \ f(x) \leq f(\bar{x}) \},$$

which is non-empty regarding Assumptions 4.1.

*Assumptions 4.1.* Let

1. $P_k$ and $Q_k$ be continuous real valued functions on $\chi = \{x \in [0,1]^n : Ao_T x = b\}$.

2. $Q_k(x)$ be positive for all $k \in P = \{1, \dots, p\}$ and all $x \in \chi$. Clearly, the generality of the problem is not to be lost by this assumption.

3. $\chi$ be a non-empty and compact set in $\mathcal{R}^n$.

4. $c_k^t$ and $d_k^t$ be the $k$-th row of $C = (c_{ij})_{p \times n}$ and $D = (d_{ij})_{p \times n}$, respectively.

5. $D \in [0,1]^{p \times n}$ and $|C| = (|c_{ij}|) \in [0,1]^{p \times n}$.

6. $C_0 = (c_1^0, \dots, c_p^0) \in [-1,1]^{p \times 1}$ and $D_0 = (d_1^0, \dots, d_p^0) \in [0,1]^{p \times 1}$.

*Definition 4.2.* Problem (4.1) is called incomplete, if there is a $k_0 \in P$ such that $d_{k_0}^t = \bar{0} \in \mathcal{R}^n$.

*Definition 4.3.* The objective functions of (4.1) are in conflict if $\chi_1^0 \cap \chi_2^0 \cap \dots \cap \chi_p^0 = \phi$, where $\chi_k^0$, $k = 1, \dots, p$, denotes the set of optimal solutions for the problem

$$\min \left\{ f_k(x) = \frac{P_k(x)}{Q_k(x)} : x \in \chi, Q_k(x) > 0 \right\},$$

and $\chi$ is a nonempty and compact set in $\mathcal{R}^n$.

Now, we deal with Theorems 4.4 – 4.5 which are similar to those of Nykowski and Zolkiewski [19].

*Theorem 4.4.* Suppose $\bar{x} \in \chi^f$.

1. If $f_k(x) = \frac{P_k(x)}{Q_k(x)} > 0$, for all $k \in P$ and all $x \in \chi$ then $\bar{x} \in \chi^\Gamma$ where

$$\chi^\Gamma = \text{argmin} \left\{ \Gamma(x) = \big( P_1(x), \dots, P_p(x), -Q_1(x), \dots, -Q_p(x) \big) : x \in \chi \right\}$$
$$= \{ \bar{x} \in [0,1]^n : \nexists x \in \chi, \Gamma(x) \leq \Gamma(\bar{x}) \}.$$

That is $\chi^f \subset \chi^\Gamma$.

2. If $f_k(x) = \frac{P_k(x)}{Q_k(x)} < 0$, for all $k \in P$ and all $x \in \chi$ then $\bar{x} \in \chi^\Lambda$ (that is $\chi^f \subset \chi^\Lambda$) where

$$\chi^\Lambda = \text{argmin}$$
$$\{ \Lambda(x) = ( P_1(x), \dots, P_p(x), Q_1(x), \dots, Q_p(x) ) : x \in \chi \}$$
$$= \{ \bar{x} \in [0,1]^n : \nexists x \in \chi, \Lambda(x) \leq \Lambda(\bar{x}) \}.$$

*Proof.* 1. Let $\bar{x} \notin \chi^\Gamma$, that is, there is an $x_0 \in \chi$ such that $\Gamma(x_0) \leq \Gamma(\bar{x})$. Then, there is a non-negative vector $d \in \mathcal{R}^{2p}$ such that $\Gamma(\bar{x}) = \Gamma(x_0) + d$. Thus,

$$f_k(\bar{x}) = \frac{P_k(\bar{x})}{Q_k(\bar{x})} = \frac{P_k(x_0) + d_k}{Q_k(x_0) - d_{p+k}} \geq \frac{P_k(x_0)}{Q_k(x_0)} = f_k(x_0),$$

$$k = 1, \dots, p.$$

Additionally, there is an $s \in P$ such that $d_s + d_{p+s} > 0$, and $f_s(\bar{x}) > f_s(x_0)$. Therefore, $f(\bar{x}) \geq f(x_0)$, that is, $\bar{x} \notin \chi^f$.

2. Let $\bar{x} \notin \chi^\Lambda$, that is, there is an $x_0 \in \chi$ such that $\Lambda(x_0) \leq \Lambda(\bar{x})$. There is a non-negative vector $d \in \mathcal{R}^{2p}$ such that $\Lambda(\bar{x}) = \Lambda(x_0) + d$. Thus, for $k \in P$,

$$f_k(\bar{x}) = \frac{P_k(\bar{x})}{Q_k(\bar{x})} = \frac{P_k(x_0) + d_k}{Q_k(x_0) + d_{p+k}} \geq \frac{P_k(x_0)}{Q_k(x_0) + d_{p+k}}.$$

$$\frac{1}{Q_k(x_0) + d_{p+k}} \leq \frac{1}{Q_k(x_0)} \text{ and } P_k(x_0) < 0$$

$$\Rightarrow \frac{P_k(x_0)}{Q_k(x_0) + d_{p+k}} \geq \frac{P_k(x_0)}{Q_k(x_0)}.$$

Therefore,

$$f_k(\bar{x}) \geq \frac{P_k(x_0)}{Q_k(x_0) + d_{p+k}} \geq \frac{P_k(x_0)}{Q_k(x_0)} = f_k(x_0).$$

Similar to part (1), it can be shown that $\bar{x} \notin \chi^f$ which is a contradiction.

*Theorem 4.5.* Suppose $\bar{x} \in \chi^f$. If $f_k(x) = \frac{P_k(x)}{Q_k(x)} > 0$, for $k \in \{1, \dots, h\}$ and $f_k(x) = \frac{P_k(x)}{Q_k(x)} < 0$, for $k \in \{h+1, \dots, p\}$ (for all $x \in \chi$) then $\bar{x} \in \chi^\Phi$ where
$\chi^\Phi = \operatorname{argmin}\{\Phi(x): x \in \chi\} = \{\bar{x} \in [0,1]^n : \nexists x \in \chi, \quad \Phi(x) \leq \Phi(\bar{x})\}$,
where

$$\Phi(x) = \left(P_1(x), \dots, P_p(x), -Q_1(x), \dots, -Q_h(x), Q_{h+1}(x), \dots, Q_p(x)\right).$$

That is $\chi^f \subset \chi^\Phi$.

*Proof.* Let $\bar{x} \notin \chi^\Phi$, that is, there is an $x_0 \in \chi$ such that $\Phi(x_0) \leq \Phi(\bar{x})$. Thus, there is a non-negative vector $d \in \mathcal{R}^{2p}$ such that $\Phi(\bar{x}) = \Phi(x_0) + d$. Similar to Theorem 4.4, the proof can be completed.

*Remark 4.6.* Theorems 4.4 and 4.5 can also be applied on an incomplete linear fractional multi-objective problem.

*Remark 4.7.* According to the mathematical programming theory, some constant values can be added to each objective function in (4.1), with no effect on the optimal solutions set. This means that $\operatorname{argmin}\{F(x) = (f_1(x) + d_1, \dots, f_p(x) + d_p): x \in \chi\} = \chi^f$ where $(d_1, \dots, d_p) \in \mathcal{R}^p$. Therefore, we are always able to consider the equivalent MOP problem with positive objective functions on $\chi$ instead of the original problem (4.1), using the following algorithm:

*Algorithm 4.8.*
1. Given MOP (4.1) and let $\varepsilon > 0$ be a small user-defined scalar, say $\varepsilon = 10^{-8}$.

2. Define a constant vector $\delta = (\delta_1, \dots, \delta_p) \in \mathcal{R}^p$ as follows:

$$\delta_k = \begin{cases} 0, & \text{If } f_k(x) > 0, \text{ for all } x \in \chi, \\ \dfrac{-\breve{P}_k}{\breve{Q}_k} + \varepsilon, & \text{otherwise,} \end{cases}$$

where $\breve{P}_k = \min_{x \in \chi} P_k(x)$ and $\breve{Q}_k = \min_{x \in \chi} Q_k(x)$, for $k \in P$.

3. The equivalent problem of (4.1) with positive objective functions on $\chi$ is

$$\min_{x \in \chi} F(x) = \left(f_1(x) + \delta_1, \dots, f_p(x) + \delta_p\right). \quad (4.2)$$

Since there are finitely many objective functions ($p < \infty$), and in the course of the algorithm, two linear programming problems each with a single objective function are solved when there is an $x \in \chi$ and a $k \in \{1, \dots, p\}$ such

that $f_k(x) < 0$, Algorithm 4.8 terminates. Moreover, the following lemma shows the efficiency of Algorithm 4.8.

*Lemma 4.9.* The objective functions of (4.2) are positive on $\chi$.

*Proof.* Consider an objective function $f_k(x)$ and an $x_0 \in \chi$ with $f_k(x_0) \leq 0$. We show that $f_k(x_0) + \delta_k > 0$.

$$f_k(x_0) + \delta_k = f_k(x_0) + \frac{-\breve{P}_k}{\breve{Q}_k} + \varepsilon$$

$$= f_k(x_0) + \frac{-\min_x P_k(x)}{\min_x Q_k(x)} + \varepsilon$$

$$= f_k(x_0) + \frac{\max_x(-P_k(x))}{\min_x Q_k(x)} + \varepsilon$$

$$\geq f_k(x_0) + \frac{-P_k(x_0)}{Q_k(x_0)} + \varepsilon$$

$$= f_k(x_0) - f_k(x_0) + \varepsilon$$

$$= \varepsilon > 0$$

*Remark 4.10.* By Theorem 4.4, the linear multi-objective problem related to (4.2) is
$$\min_{x \in \chi} \bar{\Gamma}(x) = \big(P_1(x) + \delta_1 Q_1(x), \dots, P_p(x) + \delta_p Q_p(x),$$
$$-Q_1(x), \dots, -Q_p(x)\big) \quad (4.3)$$
and we have $\chi^F \subseteq \chi^{\bar{\Gamma}}$.

Assume that the FRE system (3.1) is simplified as much as possible by the results in Section 3. Let $\chi$ be the reduced feasible set. Using Assumption 4.1, problem (4.3) can be reformulated as

$$\min_{x \in \chi} \bar{\Gamma}(x) = \Big((c_1^t + \delta_1 d_1^t)x + (c_1^0 + \delta_1 d_1^0), \dots, (c_p^t + \delta_p d_p^t)x + (c_p^0 + \delta_p d_p^0), -d_1^t x - d_1^0, \dots, -d_p^t x - d_p^0\Big). \quad (4.4)$$

By omitting the constant values of the objective functions, we have

$$\min_{x \in \chi} \bar{\bar{\Gamma}}(x) = \Big((c_1^t + \delta_1 d_1^t)x, \dots, (c_p^t + \delta_p d_p^t)x, -d_1^t x, \dots, -d_p^t x\Big). \quad (4.5)$$

Note that $\chi^{\bar{\bar{\Gamma}}} = \chi^{\bar{\Gamma}}$.

Now, we find $\chi^{\bar{\bar{\Gamma}}}$ by applying the improved $\varepsilon$-constraint method (2.4) on problem (4.5). Note that the set of efficient solutions of problem (4.1), $\chi^f$, is a subset of $\chi^{\bar{\bar{\Gamma}}}$, i.e. $\chi^f \subseteq \chi^{\bar{\bar{\Gamma}}} \subseteq \chi$. Hence, to obtain $\chi^f$, it is sufficient to solve the following problem on a smaller feasible set $\chi^{\bar{\bar{\Gamma}}}$ instead of $\chi$:

$$\min_{x \in \chi^{\bar{\Gamma}}} z = f(x) = \big(f_1(x), \dots, f_p(x)\big).$$

Since $\chi^{\bar{\bar{\Gamma}}}$ is a discrete set, a $p$ dimension vector can be related to each individual of $\chi^{\bar{\bar{\Gamma}}}$ in which the $k$-th element individually evaluates the $k$-th objective function. The efficient solutions can be obtained by comparing such vectors.

# 5. Numerical Testing

*Example 5.1.* Here, we are going to apply the proposed method in Section 4 to solve a constrained linear fractional optimization problem. Consider the following randomly

generated consistent constrained optimization problem.

$$\min f(x)$$
$$\text{s.t.}$$
$$\begin{pmatrix} 0.4588 & 0.6620 & 0.7703 & 0.3503 \\ 0.6620 & 0.4162 & 0.8419 & 0.8329 \\ 0.2564 & 0.0106 & 0.4363 & 0.4273 \\ 0.8700 & 0.2649 & 0.3181 & 0.1193 \end{pmatrix} o_{av} x = \begin{pmatrix} 0.5770 \\ 0.6786 \\ 0.4758 \\ 0.7826 \end{pmatrix},$$
$$x \in [0,1]^4,$$

where $f: \mathcal{R}^4 \to \mathcal{R}^3$, is defined as

$$f_1(x) = \frac{-0.4803x\_1 + 0.5184x\_2 + 0.1337x\_3 + 0.5843x\_4 - 0.9743}{0.8382x\_1 + 0.6911x\_2 + 0.9714 \, x\_3 + 0.6385x\_4 + 0.5679},$$

$$f_2(x) = \frac{0.7561x\_1 - 0.9366 \, x\_2 - 0.2472x\_3 - 0.7091x\_4 - 0.6268}{0.1411x\_1 + 0.0345 \, x\_2 + 0.1125x\_3 + 0.5942x\_4 + 0.4265},$$

$$f_3(x) = \frac{-0.6235x\_1 + 0.2847x\_2 - 0.5749x\_3 - 0.0217x\_4 - 0.0295}{0.7322x\_1 + 0.4889x\_2 + 0.7432x\_3 + 0.4986x\_4 + 0.0762}.$$

First, according to results in Section 3, we reduce the feasible set of the problem as much as possible. According to Theorem 3.12 and Corollary 3.21, the redundant third and fourth constraints can be ignored and the first component of solutions is fixed to 0.6952 and this component can be eliminated from the solution space as well as the first column of the coefficient matrix from consideration. Thus, the constrained linear fractional optimization problem is reduced to

$$\min \bar{f}(x)$$
$$\text{s.t.}$$
$$\begin{pmatrix} 0.6620 & 0.7703 & 0.0000 \\ 0.0000 & 0.0000 & 0.8329 \end{pmatrix} o_{av} x = \begin{pmatrix} 0.5770 \\ 0.6786 \end{pmatrix}$$
$$x \in [0,1]^3,$$

where the reduced objective function $\bar{f}: \mathcal{R}^3 \to \mathcal{R}^3$, is

$$\bar{f}_1(x) = \frac{0.5184x\_1 + 0.1337x\_2 + 0.5843x\_3 - 0.9743}{0.6911x\_1 + 0.9714 \, x\_2 + 0.6385x\_3 + 0.5679},$$

$$\bar{f}_2(x) = \frac{-0.9366 \, x\_1 - 0.2472x\_2 - 0.7091x\_3 - 0.6268}{0.0345 \, x\_1 + 0.1125x\_2 + 0.5942x\_3 + 0.4265},$$
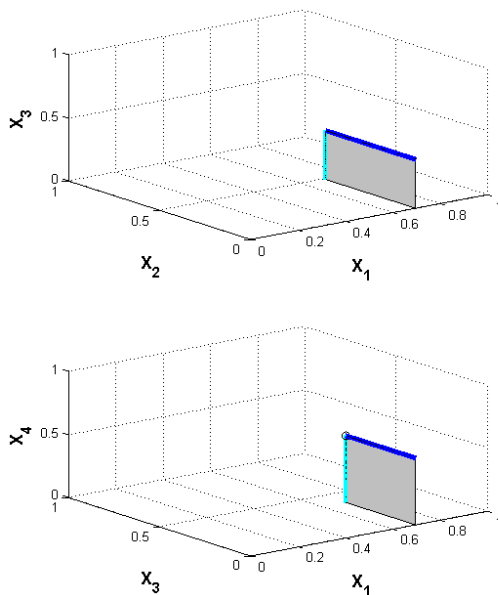
$$\bar{f}_3(x) = \frac{0.2847x\_1 - 0.5749x\_2 - 0.0217x\_3 - 0.0295}{0.4889x\_1 + 0.7432x\_2 + 0.4986x\_3 + 0.0762}.$$

The maximum solution and the minimal solutions of the feasible region are $\hat{x} = (0.4920, 0.3837, 0.5243)^t$, $\check{x}^1 = (0.4920, 0, 0.5243)^t$ and $\check{x}^2 = (0, 0.3837, 0.5243)^t$, respectively. Solving the problem following the method described in Section 4, it is concluded that

$$\bar{\chi}^f = \{\tilde{x}^1 = (0.4920, 0.0000, 0.5243)^t,$$
$$\tilde{x}^2 = (0.0000, 0.3837, 0.5243)^t\},$$

$$\bar{f}(\tilde{x}^1) = (-0.3323, -1.9329, 0.1715)$$

and

$$\bar{f}(\tilde{x}^2) = (-0.4835, -1.3997, -0.4199).$$

Therefore, by adding the first component of solutions with the value 0.6952, the original solutions are obtained as

$$\chi^f = \{\tilde{x}^1 = (0.6952, 0.4920, 0.0000, 0.5243)^t,$$
$$\tilde{x}^2 = (0.6952, 0.0000, 0.3837, 0.5243)^t\},$$

$$f(\tilde{x}^1) = (-0.4091, -1.0945, -0.3075)$$

and

$$f(\tilde{x}^2) = (-0.5116, -0.6457, -0.6140).$$

In Figures 1 and 2, different sections of the original feasible region and different sections of the original objective functions with the optimal solutions are shown, respectively.

In addition, Figures 3 and 4 show different sections of the reduced feasible region and different sections of the reduced objective functions with the optimal solutions, respectively. The fixed variables and their values are shown in each figure.
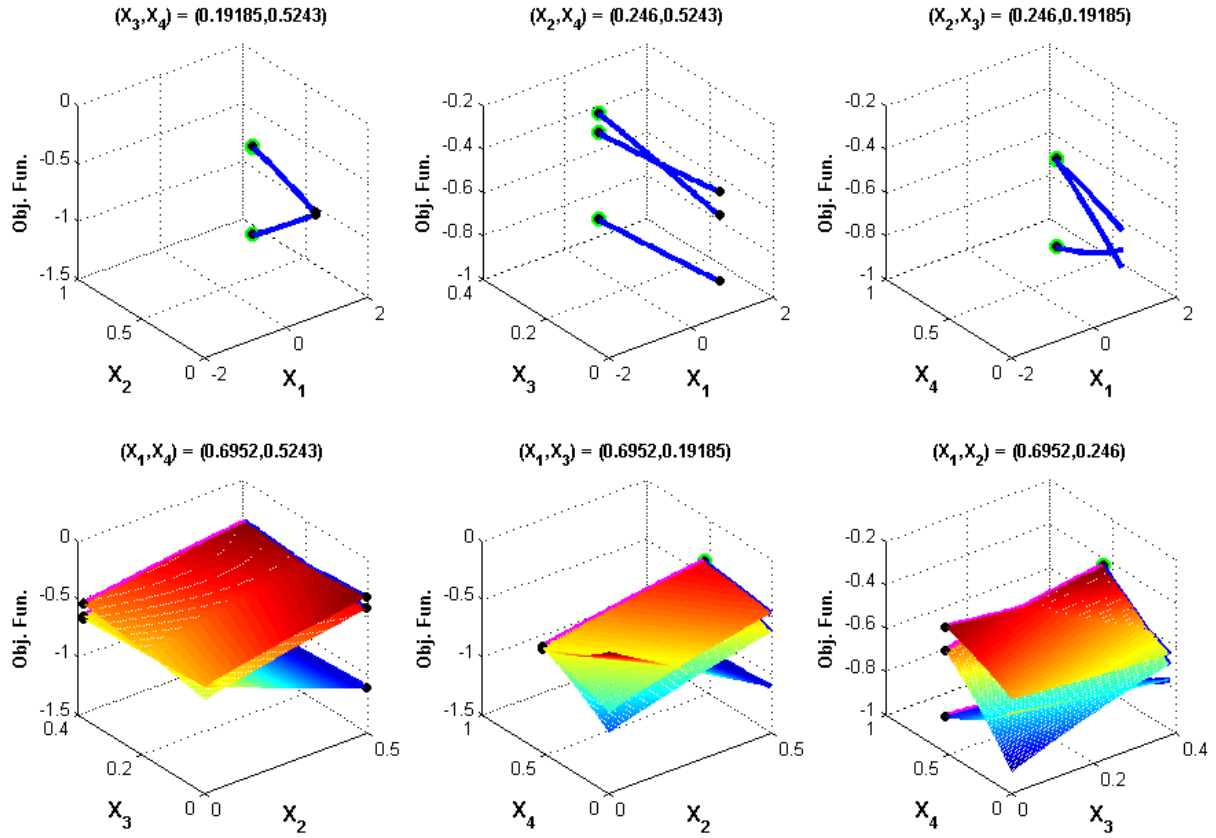


*Figure 1. The feasible space of Example 5.1*

**Figure 2.** *The objective function and the optimal solutions of Example 5.1*
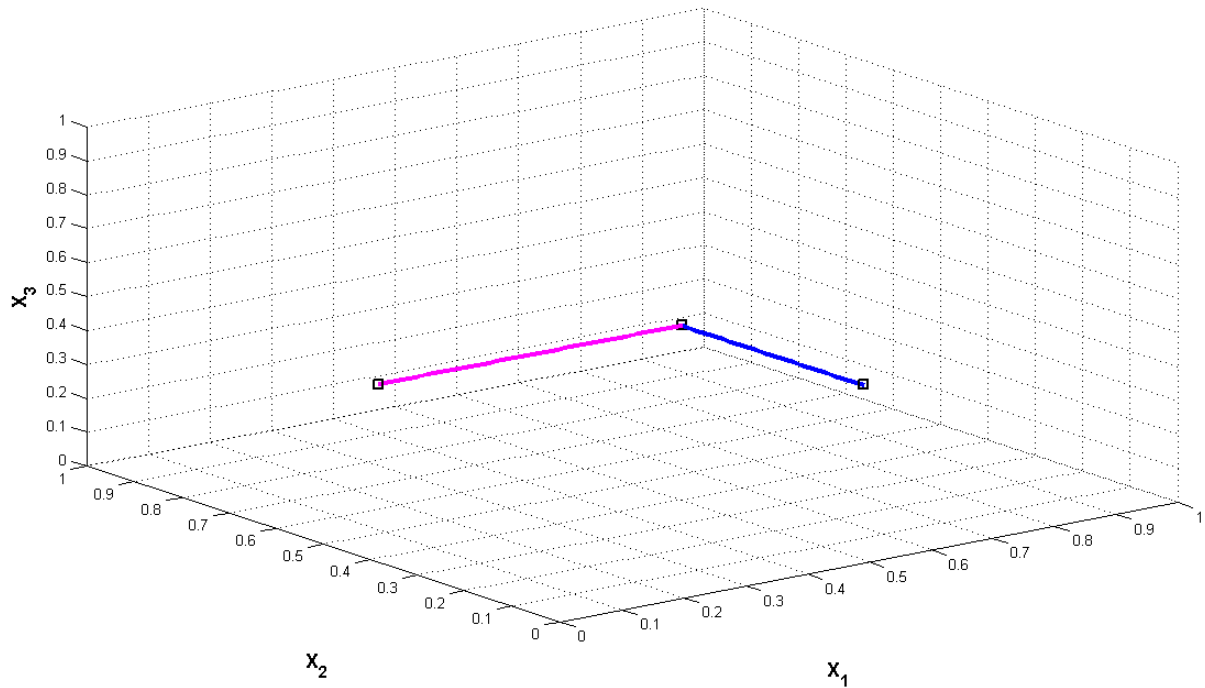


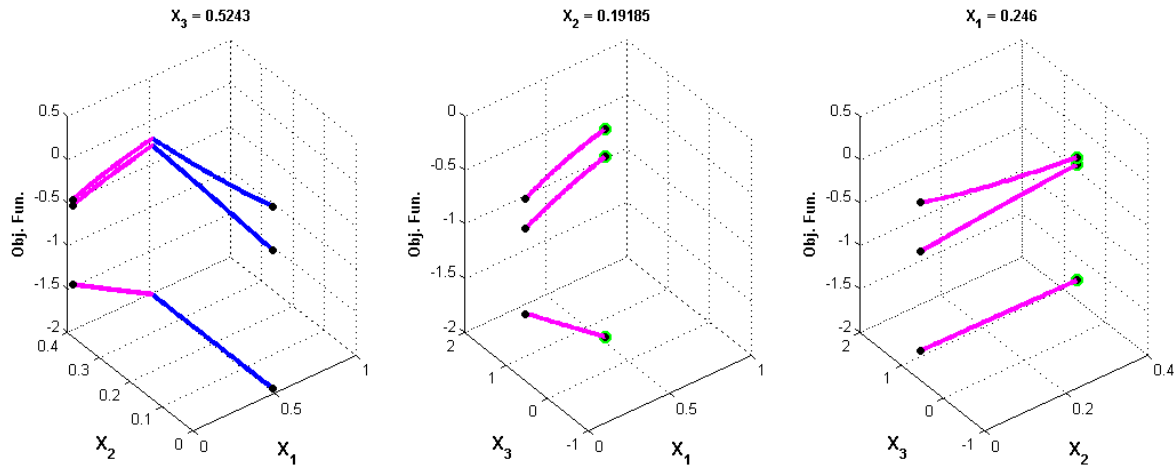**Figure 3.** *The reduced feasible space of Example 5.1*

***Figure 4.*** *The reduced objective function and the optimal solutions of Example 5.1*

# 6. Conclusions

In this paper, we obtained the efficient solutions of linear fractional multi-objective optimization problems (LFMOP) subject to a system of fuzzy relational equations (FRE) using the max-average composition. First, some theorems and results were presented to thoroughly identify and reduce the feasible set of the FRE. Then, the LFMOP was converted to a linear multi-objective optimization problem using Nykowski and Zolkiewski's approach. Finally, the efficient solutions were obtained by applying the improved $\varepsilon$ -constraint method. We tested the efficiency of the proposed method by solving a consistent test problem.

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