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Lagrange-type Algebraic Minimal Bivariate Fractal Interpolation Formula

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Abstract: Fractal interpolation methods became an important method in data processing, even for functions with abrupt changes. In the last few decades it has attracted several authors because it can be applied in various fields. The advantage of these methods are that we can generalize the classical approximation methods and also we can combine these methods for example with Lagrange interpolation, Hermite interpolation or spline interpolation. The classical Lagrange interpolation problem give the construction of a suitable approximate function based on the values of the function on given points. These method was generalized for more than one variable functions. In this article we generalize the so-called algebraic maximal Lagrange interpolation formula in order to approximate functions on a rectangular domain with fractal functions. The construction of the fractal function is made with a so-called iterated function system. This method it has the advantage that all classical methods can be obtained as a particular case of a fractal function. We also use the construction for a polynomial type fractal function and we proof that the Lagrange-type algebraic minimal bivariate fractal function satisfies the required interpolation conditions. Also we give a delimitation of the error, using the result regarding the error of a polynomial fractal interpolation function.

Keywords: Fractal Interpolation, Lagrange Interpolation, Fractal Surfaces

1. Introduction

A function $f : [a, b] \to \mathbb{R}$, defined on the real interval is named by Barnsley a *fractal function* if the Hausdorff dimension of the graph is noninteger. Barnsley introduced the notion of a fractal interpolation function (FIF) [1]. He said that a fractal function is a (FIF) if it possess some interpolation properties, it means that the fractal function is constrained to go throw on a distinct set of points $(x_i, y_i) \in \mathbb{R}^2, i = 1, 2, ..., N$. In the last few decades the methods of fractal interpolation methods was applied successfully in many fields of applied sciences. It has the advantage that it can be also combine with the classical methods or real data interpolation [4–6, 14]. Hutchinson and Rüschendorf gave the stochastic version of the fractal interpolation function [9]. In order to obtain fractal interpolation functions with more flexibility Wang and Yu use instead of a constant scaling parameter a variable vertical scaling factor [15]. Barnsley introduced the notion of local iterated function systems which are an important generalization of the global iterated function systems [3] . Massopust introduced the fractal surfaces using an iterated function system [10]. He considered the case when the domain was triangular. Than was studied the bivariate fractal functions on grids [8]. Recently was introduced a new construction of fractal interpolation functions on surfaces using the bivariate Hermite interpolation method [12]. The construction of the so-called minimal and maximal Lagrange type interpolation formula is also known [7]. Using this classical formula and the method used for the construction of a bivariate fractal interpolation function, in this paper we study the fractal version of the Lagrange-type algebraic maximal interpolation formula.

2. Fractal Interpolation Functions

Let $\{(x_i, y_i) \in R^2, i = 0, 1, \cdots, N\}$ be given, $I = [x_0, x_N]$ and $X = I \times [a, b]$ with Euclidean metric d, $I_n = [x_{n-1}, x_n]$, $u_n : I \to I_n$, $n \in \{1, 2, ..., N\}$ are contractiv homeomorphism such that

$$u_n(x_0) := x_{n-1},\ u_n(x_N) := x_n,\ \forall n \in \{1, \cdots, N\}.$$
$$|u_n(c_1) - u_n(c_2)| \leq l|c_1 - c_2|,\ c_1, c_2 \in I,\ 0 \leq l < 1$$

$v_n : X \to [a, b]$ continuous, with

$$v_n(x_0, y_0) := y_{n-1},\ v_n(x_N, y_N) := y_n,\ \forall n \in \{1, \cdots, N\}.$$
$$|v_n(c, d_1) - v_n(c, d_2)| \leq q|d_1 - d_2|,\ c \in I,\ d_1, d_2 \in [a, b], 0 \leq q < 1.$$

Let $w_n : X \to X,\ n \in \{1, 2, ..., N\}$
$$w_n(x, y) = (u_n(x), v_n(x, y)).$$

$\{X, w_n : n = 1, 2, ..., N\}$ is an iterated function system (IFS) but may not be hyperbolic. The functions $f : I \to R$, which interpolate the data according to f$(x_i) = y_i,\ i = 0, 1, ..., N$, and whose graphs are attractors of IFS are *fractal interpolation functions*

Theorem 2.1 (Barnsley) For the IFS $\{X, w_n : n = 1, 2, ..., N\}$ defined above, there is a metric d equivalent to the Euclidean metric, such that the IFS is hyperbolic with respect to d. The unique attractor G of the IFS is the graph of a continuous function $f : I \to R$ which interpolates the date set $\{(x_i, y_i) \in R^2, i = 0, 1, \cdots, N\}$

The function f is the fixed point of the Read-Bajrakterovic operator R on $C(I)$:

$$(Rg)(t) = v_n(u_n^{-1}(t), g \circ u_n^{-1}(t)), t \in I_n, n \in \mathbb{N}.$$

The fractal function f which interpolate the data set

$$\{(x_i, y_i) \in R^2, i = 0, 1, \cdots, N\}$$

is a unique function that satisfies the functional equation

$$f(t) = v_n(u_n^{-1}(t), f \circ u_n^{-1}(t)), \forall t \in I_n, n \in \mathbb{N}_N.$$

The most frequently used iterated function system is determined by the following functions

$$u_n(t) = a_n t + b_n,\ \ v_n(t, x) = \alpha_n x + q_n(t),$$

where $q_n : I \to \mathbb{R}$ are continuous functions such that

$$q_n(t) = f \circ u_n(t) - \alpha_n f \circ c(t)$$

and c is an increasing continuous function such that $c(t_0) = t_0,\ c(t_N) = t_N$. For example for the function c on the interval $[0, 1]$ we can consider the family of functions $c(t) = (e^{\lambda t} - 1)/(e^{\lambda} - 1)$.

3. Lagrange-type Algebraic Maximal Approximation Formula

Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$. We suppose that we know the value of the function f on a given set of distinct points $(x_i, y_j) \in D, i = 1, .., n,\ \ j = 1, ..., m$. If we keep y fixed we can write the following Lagrange-type approximation

$$f(x, y) = \sum_{i=0}^{n} l_i(x) f(x_i, y) + \frac{p(x)}{(n+1)!} f^{(n+1,0)}(\xi, y) \tag{1}$$

where $p(x) = \prod_{i=0}^{n}(x - x_i)$ and $\xi \in (\alpha_1, \beta_1),\ \alpha_1 = \min\{x_0, x_1, ..., x_n\},\ \beta = \max\{x_0, x_1, ..., x_n\}$.

Now let x be fixed, and we have

$$f(x, y) = \sum_{j=0}^{m} \tilde{l}_j(y) f(x, y_j) + \frac{t(y)}{(m+1)!} f^{(0,m+1)}(x, \eta) \tag{2}$$

where $t(y) = \prod_{j=0}^{m}(y - y_j)$ and $\eta \in (\alpha_2, \beta_2),\ \alpha_2 = \min\{y_0, y_1, ..., y_m\}$, and $\beta_2 = \max\{y_0, y_1, ..., y_m\}$ and $l_i,\ \tilde{l}_j$ are the well-known fundamental Lagrange-polynomials

$$l_i(x) = \frac{\prod_{k=0,k\neq i}^{n}(x - x_k)}{\prod_{k=0,k\neq i}^{n}(x_i - x_k)}, \tilde{l}_j(y) = \frac{\prod_{k=0,k\neq j}^{m}(y - y_k)}{\prod_{k=0,k\neq j}^{m}(y_j - y_k)}.$$

We use the following notations $L_n^x,\ L_m^y$ are the Lagrange interpolation operators with regard to the variable x and y, and $R_n^x,\ R_m^y$ are the corresponding remainder operators. Then the algebraic maximal approximation formula is:

$$f = L_n^x L_m^y f + R_n^x \oplus R_m^y f.$$

This interpolation formula is a discreet(punctual) interpolation since

$$(L_n^x L_m^y f)(x_i, y_j) = f(x_i, y_j),\ i = 0, ...n; j = 0, ..., m.$$

Considering the corresponding remainder terms of the formulas (1), (2) we have

$$(R_n^x \oplus R_m^y f)(x,y) = \frac{p(x)}{(n+1)!} f^{(n+1,0)}(\xi, y) + \frac{t(y)}{(m+1)!} f^{(o,m+1)}(x,\eta) - \frac{p(x)t(y)}{(n+1)!(m+1)!} f^{(n+1,m+1)}(\xi,\eta).$$

So, we have the following Lagrange-type algebraic maximal approximation formula of he function f

$$\begin{aligned} f(x,y) &= \sum_{i=0}^{n}\sum_{j=0}^{m} l_i(x)\tilde{l}_j(y) f(x_i,y_j) + \frac{p(x)}{(n+1)!}\sum_{j=0}^{m}\tilde{l}(y) f^{(n+1,0)}(\xi, y_j) \\ &+ \frac{t(y)}{(m+1)!}\sum_{i=0}^{n} l_i(x) f^{(0,m+1)} f(x_i,\eta) + \frac{p(x)p(y)}{(n+1)!(m+1)!} f^{(n+1,m+1)}(\xi,\eta). \end{aligned}$$

4. Generalization of Algebraic Maximal Lagrange Interpolation Formula by Fractal Interpolation

Let us consider a set of equidistant set of data $x_0 < x_1 < ... < x_m,\ y_0 < y_1 < ... < y_s$, and suppose that the value of the function $f(x_i, y_j), i = 0, 1, ..., m;\ j = 0, 1, ..., s$ are given.

Theorem 4.1 Let be given $a \le x_0 < x_1 < ... < x_m \le b,\ c \le y_0 < y_1 < ... < y_s \le d$, a set of equidistant data also the value of the function $f(x_i, y_j), i = 0, 1, ..., m;\ j = 0, 1, ..., s$ are given. The vertical scaling factors $\alpha_i,\ i = 1, 2, ..., m$ and $\beta_j,\ j = 1, 2, ..., s$ such that $|\alpha|_\infty, 1$, and $|\beta|_\infty < 1$. Than for a fixed $i, i = 0, 1, ..., m$ and $j, j = 0, 1, , , , .s$ there exist a fractal function L_{im}^{α} and L_{js}^{β} such that

$$L_{im}^{\alpha}(x_i) = L_{im}(x_i),\ \text{for all } i = 0, 1, ..., m \qquad (3)$$

$$L_{js}^{\beta}(y_j) = L_{js}(y_j),\ \text{for all } j = 0, 1, ..., s. \qquad (4)$$

Proof We proof the conditions of the theorem of Barnsley for u_n, v_n. The function u_n is a contractive homeomorphism and $u_n(x_0) = x_{n-1},\ u_n(x_m) = x_n$ and $L_m^x(x_n) = f_n, n = 0, 1, ..., m$.

$$\begin{aligned} v_n(x_0, f_0) &= \alpha_n f_0 + q_n(x_0) \\ &= \alpha_n f_0 + L_m^x \circ u_n(x_0) - \alpha_n L_m^x \circ c(x_0) \\ &= \alpha f_0 + L_m^x(x_{n-1}) - \alpha f_0 = f_{n-1} \end{aligned}$$

$$\begin{aligned} v_n(x_m, f_m) &= \alpha_n f_m + q_n(x_m) \\ &= \alpha_n f_m + L_m^x \circ u_n(x_m) - \alpha_n L_m^x \circ c(x_m) \\ &= \alpha f_m + L_m^x(x_n) - \alpha f_m = f_n \end{aligned}$$

We will consoder the Read-Bajrakterovic operator defined on $C([a,b]x[c,d])$ regarding with the variable x

$$R_\alpha f(\cdot, y) = v_n(u_n^{-1}(t), f \circ u_n^{-1}(x).$$

This is a contraction on the second variable , therefore it has a unique fixed point, $L_m^{x\alpha}$.

$$L_m^{x\alpha}(x) = \alpha_n L_m^{x\alpha} \circ u_n^{-1}(x) + q_n \circ u_n^{-1}(x),\ x \in I_n,$$

$$L_m^{x\alpha}(x) = L_m^x + \alpha_n(L_m^{x\alpha} - L_m^x \circ c) \circ u_n^{-1}(x),\ x \in I_n.$$

$L_m^{x\alpha}$ interpolates the points (x_n, f_n):

$$\begin{aligned} L_m^{x\alpha} &= v_n(u_n^{-1}(x_n), L_m^{x\alpha} \circ u_n^{-1}(x_n)) \\ &= v_n(x_m, L_m^{x\alpha}(x_m)) = v_n(x_m, f_m) = f_n. \end{aligned}$$

$L_i m^{x\alpha}$ is the corresponding Lagrange type fractal function in the direction x. In the same way we can give the construction of the fractal function $L_s^{y\beta}$ im the y direction. The IFS for $L_{js}^{y\beta}$ is given by $\{\mathbb{R}^2, \omega_m^y : s = 1, 2, ..., s\}$, were

$$\begin{aligned} u_j s^y(t) &= a_j t + b_j \\ v_j s^y(t,x) &= \beta_j x + L_{js}(u_{js}(t)) - \beta_j L_{js}(c(t)) \end{aligned}$$

where $a_j = (y_j - y_{j-1})/(y_s - y_0), b_j = (y_s y_{j-1} - y_0 y_j)/(y_s - y_0)$ and c is an increasing continuous function which satisfies the following conditions: $c(y_0) = y_0, c(y_s) = y_s$.

Definition 4.1 The Lagrange-type algebraic minimal bivariate fractal interpolation function is given by

$$L_{ms}^{\alpha,\beta} f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{s} L_{im}^{\alpha}(x) L_{js}^{\beta}(y) f(x_i, y_j)$$

Remark 4.1 When the vertical scaling vectors $\alpha = 0$ and $\beta = 0$ we obtain the classical bivariate algebraic maximal Lagrange interpolation function.

5. Delimitation of the Error

Theorem 5.1 Let $f \in C([a,b] \times [c,d])$ be the function approximated by the Lagrange-type algebraic minimal bivariate fractal interpolation function $L_{MN}^{\alpha,\beta}$, where the vertical scaling parameters $\alpha,\ \beta$ are satisfying $\|\alpha\|_\infty < 1,\ \|\beta\|_\infty < 1$, then

$$\| \quad f - L^{\alpha,\beta}_{MN} f\|_\infty \leq \|f - L_{MN}\|_\infty + \|f\|_\infty \sum_{m=0}^{M}\sum_{n=0}^{N}\left(\frac{2\|\alpha\|_\infty}{1-\|\alpha\|_\infty} + \frac{2\|\beta\|_\infty}{1-\|\beta\|_\infty}\right)\|l_m\|_\infty \|ln\|_\infty$$

Proof

$$\|f - L^{\alpha\beta}_{MN}\|_\infty \leq \|f - L_{MN}\|_\infty + \|L_{MN} - L^{\alpha\beta}_{MN}\|_\infty$$

$$\begin{aligned}
\|L_{MN} \quad &- \quad L^{\alpha\beta}_{MN}\|_\infty = \max_{(x,y)\in[a,b]\times[c.d]} |L_{MN}(x,y) - L^{\alpha\beta}_{MN}(x,y)| \\
&= \max|\sum_{m=0}^{M}\sum_{n=0}^{N} l_m(x)l_n(y)f(x_M,y_n) - \sum_{m=0}^{M}\sum_{n=0}^{N} l^\alpha_m(x)l^\beta_n(y)f(x_M,y_n)| \\
&\leq \max\sum_{m=0}^{M}\sum_{n=0}^{N} |l_m(x)l_n(y) - l^\alpha_m(x)l^\beta_n(y)|\cdot\|f\|_\infty \\
&\leq \max\sum_{m=0}^{M}\sum_{n=0}^{N} |l_m(x)l_n(y) - l^\alpha_m(x)l_n(y) + l^\alpha_m(x)l_n(y) - l^\alpha_m(x)l^\beta_n(y)|\cdot\|f\|_\infty \\
&\leq \sum_{m=0}^{M}\sum_{n=0}^{N}\left(\|l_m - l^\alpha_m\|_\infty\cdot\|l_m\|_\infty + + \|l_m\|_\infty\cdot\|l_n - l^\beta_n\|_\infty\right)\|f\|_\infty \\
&\leq \|f\|_\infty\sum_{m=0}^{M}\sum_{n=0}^{N}\left(\frac{2\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|l_m\|_\infty\|ln\|_\infty + + \frac{2\|\beta\|_\infty}{1-\|\beta\|_\infty}\|l_m\|_\infty\|ln\|_\infty\right) \\
&= \|f\|_\infty\sum_{m=0}^{M}\sum_{n=0}^{N}\left(\frac{2\|\alpha\|_\infty}{1-\|\alpha\|_\infty} + \frac{2\|\beta\|_\infty}{1-\|\beta\|_\infty}\right)\|l_m\|_\infty\|l_n\|_\infty,
\end{aligned}$$

where in the last inequality we use the result form regarding the error of a fractal interpolation function [11]. If f^α is the fractal interpolation function of a polinomial function f, than the uniform disatnce between them verifies

$$\|f^\alpha - f\|_\infty \leq \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f\|_\infty.$$

6. Conclusion

The present paper give a method to construct a bivariate Lagrange-type fractal function in the case of a rectangular domain. The advantage of these method is that it is easy to construct and to implement for the approximation of different shapes. The constructed shape can be modified by the scaling vectors, in this way we can obtain various surfaces for the graph of the FIF. So it can be a very effective tool in computer graphics, data visualization and CAGD.

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