

# A $q$ -Operational Equation for Carlitz's $q$ -Operators with Some Applications

Jian Cao<sup>1</sup>, Cheng Zhang<sup>1</sup>, Sama Arjika<sup>2,\*</sup>

<sup>1</sup>School of Mathematics, Hangzhou Normal University, Hangzhou, China

<sup>2</sup>Department of Mathematics and Informatics, University of Agadez, Agadez, Niger

## Email address:

21caojian@hznu.edu.cn (Jian Cao), hznuzhangcheng@163.com (Cheng Zhang), rjksama2008@gmail.com (Sama Arjika)

\*Corresponding author

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**Abstract:** Rogers–Szegő polynomials are the basis in the Scheme of basic hypergeometric orthogonal polynomials. By solving a  $q$ -operational equation with formal power series, Liu introduced a new  $q$ -exponential operational identity and developed a systematic method to prove the identities involving the Rogers–Szegő polynomials. In this paper, motivated by Carlitz's  $q$ -operators and Liu's  $q$ -operational equation, we construct an  $q$ -operational equation for Carlitz's  $q$ -operators and give some applications to some generating functions for Rogers–Szegő polynomials and Hahn polynomials, which generalize the method of exponential operator decomposition introduced by Cao and provide a new proof of results of Carlitz and Saad *et al.*. We chose Mehler's formula,  $q$ -Nielsen's formula for Rogers–Szegő polynomials and Mehler's formula for Hahn polynomials as examples to show that the  $q$ -series theory can be applied, which takes us quickly the results. One of the main characteristics of this method is that it provides an effective approach to calculate generating functions for some  $q$ -polynomials. This method also brings a new research perspective to problems of the sum and integration of  $q$ -polynomials.

**Keywords:**  $q$ -Operational Equation, Carlitz's  $q$ -operators, Rogers–Szegő Polynomials, Hahn Polynomials, Generating Function

## 1. Introduction

Here, we give some standard notations and definitions [14]. For any complex numbers  $a$  and  $q$  such that  $|q| < 1$ , we define the  $q$ -shifted factorials  $(a; q)_n$  as

$$(a; q)_0 = 1, (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

For any integer number  $m$ , we also adopt the following compact notation for the multiple  $q$ -shifted factorial

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_m; q)_n,$$

where  $n$  is a nonnegative integer or positive infinity. We will use frequently the following identity:

$$\left(\frac{q}{a}; q\right)_n = (-a)^{-n} q^{\binom{n+1}{2}} \frac{(aq^{-n}; q)_\infty}{(a; q)_\infty} = (-a)^{-n} q^{\binom{n+1}{2}} (aq^{-n}; q)_n. \quad (1)$$

The basic (or  $q$ -) hypergeometric series is defined as follows [18, 26, 27, 28, 29]:

$${}_r\Phi_s \left[ \begin{matrix} a_1, a_2, \dots, a_r; \\ b_1, b_2, \dots, b_s; \end{matrix} q; z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q, b_1, b_2, \dots, b_s; q)_n} \left[ (-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n. \quad (2)$$

The  $q$ -binomial theorem is as follows

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_{\infty}}{(z; q)_{\infty}}, \quad |z| < 1. \quad (3)$$

which can derive the following two identities

$$\exp_q(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}}, \quad |z| < 1 \quad (4)$$

and

$$\text{Exp}_q(z) \equiv \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} z^n = (-z; q)_{\infty}. \quad (5)$$

Carlitz [9], defined two  $q$ -shifted operators  $\mathbb{E}_x$  and  $\Delta_x$  as:

$$\mathbb{E}_x f(x) = f(qx),$$

and

$$\Delta_x^n f(x) := (1 - \mathbb{E}_x)(q - \mathbb{E}_x) \cdots (q^{n-1} - \mathbb{E}_x) f(x). \quad (6)$$

He also obtained the following results:

$$\mathbb{E}_x^n f(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \Delta_x^k f(x) \quad (7)$$

and

$$(x + \mathbb{E}_x)^n f(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k x^{n-k} \Delta_x^k f(x). \quad (8)$$

Recently, Cao [4] introduced the dual operator of (8) as

$$(x + \mathbb{E}_x^{-1})^n f(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)} x^{n-k} f(q^{-k}x), \quad (9)$$

where  $\mathbb{E}_x^{-1} f(x) = f(q^{-1}x)$ .

In this paper, we will use the operator  $\theta_x$  defined by [13].

$$\theta_x \{f(x)\} := \frac{\mathbb{E}_x^{-1} f(x) - f(x)}{q^{-1}x}, \quad (10)$$

and investigated the Rogers–Szegő polynomials  $g_n(x|q)$  related to theta function [8]

$$g_n(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k, \quad (11)$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} = \frac{(q^{-n}; q)_k}{(q; q)_k} (-1)^k q^{nk - \binom{k}{2}} \quad (12)$$

is denoted the  $q$ -Gaussian coefficient.

The  $q$ -Leibnitz rule for  $\theta_x$  is the following identity [13]

$$\begin{aligned} \theta_x^n \{f(x)g(x)\} \\ = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_x^k \{f(x)\} \theta_x^{n-k} \{g(xq^{-k})\}. \end{aligned} \quad (13)$$

where  $\theta_x^0$  is understood as the identity. Upon multiplying the two series in (5), we find the generating function of the polynomials  $g_n(x|q)$

$$\sum_{n=0}^{\infty} g_n(x|q) \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} = (-t, -tx; q)_{\infty}. \quad (14)$$

Using the fact that  $\theta_x(-tx; q)_{\infty} = t(-tx; q)_{\infty}$  and applying  $k$  times the operator  $\theta_x$  on both sides of equation (14), we gain:

$$\begin{aligned} \sum_{m=0}^{\infty} \theta_x^k g_m(x|q) \frac{q^{\binom{m}{2}} t^m}{(q; q)_m} &= \theta_x^k (-t, -tx; q)_{\infty} \\ &= t^k (-t, -tx; q)_{\infty} \\ &= \sum_{m=0}^{\infty} g_m(x|q) \frac{q^{\binom{m}{2}} t^{m+k}}{(q; q)_m}. \end{aligned} \quad (15)$$

Upon equating coefficients of  $t^m$  on both sides of equation (15), we get the following:

$$\theta_x^k g_m(x|q) = q^{\binom{k+1}{2} - mk} \frac{(q; q)_m}{(q; q)_{m-k}} g_{m-k}(x|q). \quad (16)$$

The polynomials (11) have the following Rogers formula [8]

$$\begin{aligned} \sum_{n,m=0}^{\infty} g_{n+m}(x|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \\ = \frac{(s, t, sx, tx; q)_{\infty}}{(stx/q; q)_{\infty}}. \end{aligned} \quad (17)$$

There are many experts studied Carlitz's  $q$ -operators and related  $q$ -operator equations. Al-Salam [2] derived some representations for the  $q$ -Laguerre polynomials comparable with the Rodrigues formula by Carlitz's  $q$ -operators. Khan [17] dealt with  $q$ -analogues of certain operational formulae for Laguerre polynomials. Cigler [11, 12] gave short and systematic proofs of many  $q$ -identities by a consistent use of operator methods. The authors of [25] have utilized Cauchy operator to prove some identities related to homogeneous Rogers–Szegő polynomials. Bengocheaa, Verde-Starb and Ortigueira [3] showed that several types of differential equations that involve  $q$ -derivatives, Fibonacci derivatives, and other Ward's derivatives, can be solved by an algebraic operational method that does not use integrals nor integral transforms [2, 3, 6, 7, 11, 12, 16, 17, 19, 20, 24, 25].

Recently, Liu [22] proved the following results and derived several  $q$ -identities.

*Proposition 1.1* ([22]). If  $f(x)$  is a function of  $x$ , then, under

suitable convergence conditions, we have the following  $q$ -exponential operational identity:

$$\exp_q(t\Delta_x)f(x) = \frac{1}{(xt; q)_\infty} \sum_{n=0}^{\infty} \frac{1}{(q; q)_n} f(q^n x), \quad (18)$$

where  $\Delta_x = x + \mathbb{E}_x$ .

Motivated by Liu's [22] method and according to the  $\theta_x$ , our aim is to establish the following result for Carlitz's  $q$ -operators and give some applications.

**Theorem 1.1.** Let  $f(x)$  and  $F(x, t)$  be a one variable and a two variable analytic function in the neighborhood of  $x = 0 \in \mathbb{C}$

$\mathbb{C}$  and  $(x, t) = (0, 0) \in \mathbb{C}^2$ . The function  $F(x, t)$  satisfies

$$\begin{cases} \theta_t F(x, t) &= \tilde{\Delta}_x F(x, t) \\ F(x, 0) &= f(x), \quad \tilde{\Delta}_x = x + \mathbb{E}_x^{-1}, \end{cases} \quad (19)$$

if and only if, the following assertion holds true:

$$\begin{aligned} \text{Exp}_q(t\tilde{\Delta}_x)f(x) &:= F(x, t) \\ &= (-xt; q)_\infty \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} f(q^{-n} x), \end{aligned} \quad (20)$$

provided under suitable convergence conditions.

**Theorem 1.2.** Let  $f(x)$  be a one variable function in the neighborhood of  $x = 0 \in \mathbb{C}$ . The assertion (21) holds true:

$$\text{Exp}_q(t\tilde{\Delta}_x)f(x) = (-t, -xt; q)_\infty \sum_{n=0}^{\infty} \frac{(xtq^{-1})^n}{(q; q)_n} \theta_x^n f(x), \quad (21)$$

provided under suitable convergence conditions.

The Theorem 1.2 allow us to give the  $q$ -version of the Burchnell formula. The formula (21) is different from Liu's [22] concluding remarks.

**Theorem 1.3.** The  $q$ -version of the Burchnell formula for the Rogers–Szegő polynomials  $g_n(x|q)$  is given by

$$\tilde{\Delta}_x^n f(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2} - k(n+1)} x^k g_{n-k}(x|q) \theta_x^k f(x). \quad (22)$$

**Theorem 1.4.** Let  $f(x, y)$  be a two variables analytic function in the neighborhood of  $(x, y) = (0, 0) \in \mathbb{C}^2$ . Then, the following assertion holds true:

$$\text{Exp}_q(t\tilde{\Delta}_x\tilde{\Delta}_y)f(x, y) = (-xyt; q)_\infty \sum_{n, m, k=0}^{\infty} \frac{q^{\binom{n}{2} + \binom{m}{2} + \binom{k}{2} - nm} x^m y^n t^{n+m+k}}{(q; q)_n (q; q)_m (q; q)_k} f(q^{-n-k} x, q^{-m-k} y), \quad (23)$$

provided under suitable convergence conditions.

Before proving Theorem 1.4, we have the following results.

**Corollary 1.1.** For  $|axy t/q| < 1$ , we have:

$$\text{Exp}_q(-t\tilde{\Delta}_x\tilde{\Delta}_y)(ax, by; q)_\infty = \frac{(xyt, ax, by, y t; q)_\infty}{(axy t/q; q)_\infty} \sum_{n, m=0}^{\infty} \frac{(q/ax; q)_n (q/y t; q)_m}{(q; q)_n (q; q)_m} (y t a^{-1})^m \frac{(q/by; q)_{n+m} b^{n+m}}{(q^2/axy t; q)_{n+m}}. \quad (24)$$

**Corollary 1.2.** For  $|xy t^2/q| < 1$ , we have:

$$\text{Exp}_q(-t\tilde{\Delta}_x\tilde{\Delta}_y) \cdot \{1\} = \frac{(t, xt, yt, xy t; q)_\infty}{(xy t^2/q; q)_\infty}. \quad (25)$$

**Remark 1.1.** Taking  $f(x, y) = (ax, by; q)_\infty$  in Theorem 1.4, equation (23) reduces to (24). Setting  $(a, b) = (0, 0)$  in Corollary 1.1, equation (24) reduces to (25).

The rest of our investigation is organized as follows. In section 2, we will give the proof of  $q$ -operational equation, which is our fundamental result, by the method of  $q$ -exponential operational identity. In section 3, we will deduce  $q$ -exponential operational identity of analytic function. In section 4, we will reprove Mehler's formula for Rogers–Szegő polynomials by  $q$ -exponential operational identity. In

section 5, we will derive  $q$ -Nielsen's formulas for Rogers–Szegő polynomials by  $q$ -exponential operational identity. In section 6, we will gain Mehler formula for the Hahn polynomials by  $q$ -exponential operational identity.

## 2. Proof of Theorems 1.1 and 1.2

Before proving Theorems 1.1 and 1.2, we need the following lemmas.

*Lemma 2.1.* We have:

$$\text{Exp}_q(-t\tilde{\Delta}_x) \cdot \left\{ \frac{(ax; q)_\infty}{(bx; q)_\infty} \right\} = \frac{(xt, ax; q)_\infty}{(bx; q)_\infty} {}_1\Phi_1 \left[ \begin{matrix} \frac{q}{ax}; \\ \frac{q}{bx}; \end{matrix} q; \frac{at}{b} \right] \quad (26)$$

and

$$\text{Exp}_q(-t\tilde{\Delta}_x) \cdot \{(ax, bx; q)_\infty\} = (xt, ax, bx; q)_\infty {}_2\Phi_2 \left[ \begin{matrix} \frac{q}{ax}, \frac{q}{bx}; \\ 0, 0; \end{matrix} q; \frac{abtx^2}{q^2} \right]. \quad (27)$$

*Proof of Lemma 2.1.* Upon setting  $f(x) = (ax; q)_\infty / (bx; q)_\infty$  in equation (20), we have:

$$\text{Exp}_q(-t\tilde{\Delta}_x) \left\{ \frac{(ax; q)_\infty}{(bx; q)_\infty} \right\} = (xt; q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} \frac{(axq^{-n}; q)_\infty}{(bxq^{-n}; q)_\infty} = \frac{(xt, ax; q)_\infty}{(bx; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} \frac{(axq^{-n}; q)_n}{(bxq^{-n}; q)_n}.$$

Using (1) in the above equation and after simplification, the right-hand side of the above equation reads:

$$\frac{(xt, ax; q)_\infty}{(bx; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (q/ax; q)_n}{(q, q/bx; q)_n} \left( \frac{at}{b} \right)^n,$$

which is the right-hand side of equation (26).

In the same way, letting  $f(x) \equiv (ax, bx; q)_\infty$  in equation (20), we have:

$$\text{Exp}_q(-t\tilde{\Delta}_x) \{(ax, bx; q)_\infty\} = (xt; q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} (axq^{-n}, bxq^{-n}; q)_\infty. \quad (28)$$

Next, using equation (1), we gain:

$$\begin{aligned} & \text{Exp}_q(-t\tilde{\Delta}_x) \{(ax, bx; q)_\infty\} \\ &= (xt; q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} (ax, bx; q)_\infty (axq^{-n}, bxq^{-n}; q)_n \\ &= (xt, ax, bx; q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} (abx^2)^n q^{-2n-2\binom{n}{2}} (q/ax, q/bx; q)_n. \end{aligned}$$

After simplification, we gain:

$$\text{Exp}_q(-t\tilde{\Delta}_x) \{(ax, bx; q)_\infty\} = (xt, ax, bx; q)_\infty \times \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} (q/ax, q/bx; q)_n}{(q; q)_n} \left( \frac{abtx^2}{q^2} \right)^n,$$

which is the desire result. This achieves the proof of Lemma 2.1.

As particular cases of the above result, we have (29), (30) and (31).

*Corollary 2.1.* We have:

$$\text{Exp}_q(-t\tilde{\Delta}_x) \cdot \{1\} = (t, xt; q)_\infty, \quad (29)$$

$$\text{Exp}_q(-s\tilde{\Delta}_x) \cdot \{(tx; q)_\infty\} = \frac{(s, tx, sx; q)_\infty}{(stx/q; q)_\infty}, \quad (30)$$

$$\text{Exp}_q(-t\tilde{\Delta}_x) \cdot \left\{ \frac{1}{(ax; q)_\infty} \right\} = \frac{(xt; q)_\infty}{(ax; q)_\infty} {}_0\Phi_1 \left[ \begin{matrix} -; \\ \frac{q}{ax}; \end{matrix} q; \frac{tq}{ax} \right]. \quad (31)$$

*Lemma 2.2.* For  $|xst/q| < 1$ , we have:

$$\operatorname{Exp}_q(-s\tilde{\Delta}_x)\operatorname{Exp}_q(-t\tilde{\Delta}_x) \cdot \{1\} = \frac{(t, s, sx, tx; q)_\infty}{(stx/q; q)_\infty}. \quad (32)$$

*Proof of Lemma 2.2.* Using the relation (29) and (30), respectively, we have:

$$\begin{aligned} \operatorname{Exp}_q(-s\tilde{\Delta}_x)\operatorname{Exp}_q(-t\tilde{\Delta}_x) \cdot \{1\} &= \operatorname{Exp}_q(-s\tilde{\Delta}_x) \cdot \{(t, tx; q)_\infty\} \\ &= (t; q)_\infty \operatorname{Exp}_q(-s\tilde{\Delta}_x) \cdot \{(tx; q)_\infty\} \\ &= \frac{(t, s, sx, tx; q)_\infty}{(stx/q; q)_\infty}. \end{aligned}$$

This completes the proof of Lemma 2.2.

Before proving Theorem 1.1, we also need the properties of complex-valued functions of several complex variables Proposition 2.1 and Proposition 2.2.

In the rest of section, we will give the fundamental theorem of our investigation. As an application, several known results in the literature are proved by the perspective of Carlitz's  $q$ -exponential operational identity.

*Proposition 2.1* ([15]). If a complex-valued function is

holomorphic (analytic) in each variable separately in an open domain  $\mathbb{D} \subset \mathbb{C}^n$ , then it is holomorphic (analytic) in  $\mathbb{D}$ .

*Proposition 2.2* ([23]). If  $f(x_1, x_2, \dots, x_k)$  is analytic at the origin  $(0, 0, \dots, 0) \in \mathbb{C}^k$ , then  $f$  can be expanded in an absolutely convergent power series as follows:

$$f(x_1, x_2, \dots, x_k) = \sum_{n_1, n_2, \dots, n_k=0}^{\infty} \alpha_{n_1, n_2, \dots, n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}.$$

*Proof of Theorem 1.1.* On one hand, from the Hartogs Theorem (see Proposition 2.1) and since  $F(x, t)$  is analytic function in a neighborhood of  $(x, t) = (0, 0) \in \mathbb{C}^2$ , there exists a sequence  $\{A_n\}$  depending on  $x$  such that

$$F(x, t) = \sum_{n=0}^{\infty} A_n(x) t^n. \quad (33)$$

Next, if we substitute equation (33) into (19) and by the fact that

$$\theta_t t^n = q^{-n+1} (1 - q^n) t^{n-1},$$

we get:

$$\sum_{n=1}^{\infty} A_n(x) q^{-n+1} (1 - q^n) t^{n-1} = \sum_{n=0}^{\infty} \tilde{\Delta}_x A_n(x) t^n. \quad (34)$$

Equating coefficients of  $t^{n-1}$ ,  $n \geq 1$  on both sides of equation (34), we have

$$A_n(x) = \frac{q^{n-1}}{1 - q^n} \tilde{\Delta}_x A_{n-1}(x).$$

By iteration, we gain:

$$A_n(x) = \frac{q^{\binom{n}{2}}}{(q; q)_n} \tilde{\Delta}_x^n A_0(x). \quad (35)$$

Setting  $F(x, 0) = A_0(x) = f(x)$  in equation (33) and using (43), we get:

$$\begin{aligned} A_n(x) &= \frac{q^{\binom{n}{2}}}{(q; q)_n} \tilde{\Delta}_x^n f(x) \\ &= \frac{q^{\binom{n}{2}}}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^{n-k} f(q^{-k}x). \end{aligned} \quad (36)$$

Now, substituting the relation (36) into (33), we get

$$F(x, t) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^{n-k} f(q^{-k} x). \quad (37)$$

After simplification, we obtain

$$F(x, t) = \sum_{n=0}^{\infty} \sum_{n=k}^{\infty} \frac{q^{\binom{n}{2}} t^n q^{k(k-n)} x^{n-k}}{(q; q)_k (q; q)_{n-k}} f(q^{-k} x). \quad (38)$$

Changing  $n \rightarrow n + k$  in the right-hand side of equation (38), we get:

$$F(x, t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^{\binom{n+k}{2}} t^{n+k} q^{-kn} x^n}{(q; q)_k (q; q)_n} f(q^{-k} x)$$

or

$$\begin{aligned} F(x, t) &= \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} f(q^{-n} x) \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} (xt)^k}{(q; q)_k} \\ &= (-xt; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} f(q^{-n} x), \end{aligned}$$

which is the desire result.

On the other hand, let us start by the right-hand side of equation (20).

$$F(x, t) = (-xt; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} f(q^{-n} x). \quad (39)$$

Next, replace  $t$  by  $q^{-1}t$ . Then, using the identity  $q^{-n} = 1 + q^{-n}(1 - q^n)$ , we get:

$$\begin{aligned} &F(x, q^{-1}t) \\ &= (-q^{-1}xt; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} q^{-n} t^n}{(q; q)_n} f(q^{-n} x) \\ &= (-q^{-1}xt; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} [1 + q^{-n}(1 - q^n)] t^n}{(q; q)_n} f(q^{-n} x) \\ &= (-q^{-1}xt; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} f(q^{-n} x) \\ &\quad + (-q^{-1}xt; q)_{\infty} \sum_{n=1}^{\infty} \frac{q^{\binom{n}{2}} q^{-n} t^n}{(q; q)_{n-1}} f(q^{-n} x). \end{aligned} \quad (40)$$

Changing the order of summation in the second term of (40), it becomes:

$$\begin{aligned} &(-q^{-1}xt; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}} q^{-n-1} t^{n+1}}{(q; q)_n} f(q^{-n-1} x) \\ &= q^{-1}t (-q^{-1}xt; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} f(q^{-n-1} x) \\ &= q^{-1}t \mathbb{E}_x^{-1} F(x, t). \end{aligned} \quad (41)$$

Putting equation (41) in the right-hand side of equation (40), we have:

$$\begin{aligned} F(x, q^{-1}t) &= (1 + q^{-1}xt)(-xt; q)_\infty \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} f(q^{-n}x) + q^{-1}t \mathbb{E}_x^{-1} F(x, t) \\ &= (1 + q^{-1}xt)F(x, t) + q^{-1}t \mathbb{E}_x^{-1} F(x, t). \end{aligned} \quad (42)$$

Making use equation (42), we have:

$$\begin{aligned} F(x, q^{-1}t) - F(x, t) &= (1 + q^{-1}xt)F(x, t) + q^{-1}t \theta_x F(x, t) - F(x, t) \\ &= q^{-1}t(x + \mathbb{E}_x^{-1})F(x, t) \\ &= q^{-1}t \tilde{\Delta}_x F(x, t) \end{aligned}$$

which implies that

$$\theta_t F(x, t) = \tilde{\Delta}_x F(x, t), \quad t \neq 0.$$

Taking  $t = 0$  in (39), we have:

$$F(x, 0) = f(x).$$

According to the above calculations, we find that the function  $F(x, t)$  satisfies the following  $q$ -difference equations

$$\begin{aligned} \theta_t F(x, t) &= \tilde{\Delta}_x F(x, t) \\ F(x, 0) &= f(x). \end{aligned}$$

This achieves the proof of Theorem 1.1.

Now, let us go back to the expansion into power series of  $(-xt; q)_\infty$  and equating the coefficients in both sides of (20), we get the following  $q$ -identity.

**Lemma 2.3.** For any non-negative integer  $n$  and if  $f$  is a function of  $x$ , we have the following assertion:

$$\tilde{\Delta}_x^n f(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^{n-k} f(q^{-k}x), \quad (43)$$

where  $\tilde{\Delta}_x = x + \mathbb{E}_x^{-1}$ .

By a direct computation, we obtain the relation

$$(\mathbb{E}_q^{-1})^n f(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2} - k(n+1)} x^k \theta_x^k f(x). \quad (44)$$

Now, we are in position to prove Theorem 1.2.

**Proof of Theorem 1.4.** Let  $f(x, y)$  be a two variable analytic function in the neighborhood of  $(x, y) = (0, 0) \in \mathbb{C}^2$ . Using equation (20), we obtain:

$$\text{Exp}_q(t \tilde{\Delta}_x) f(x, y) = (-xt; q)_\infty \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} f(q^{-n}x, y). \quad (47)$$

Next, replacing  $t$  by  $t \tilde{\Delta}_y$  in the equation (47), we get:

$$\text{Exp}_q(t \tilde{\Delta}_x \tilde{\Delta}_y) f(x, y) = \text{Exp}_q(xt \tilde{\Delta}_y) \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} \tilde{\Delta}_y^n f(q^{-n}x, y). \quad (48)$$

**Proof of Theorem 1.2.** Putting (44) in the right-hand side of (20), we get the desired result.

Upon setting  $f(x) = 1$  in the relation (43) and for non-negative integers  $n$  and  $m$ , we obtain the identities

$$\tilde{\Delta}_x^n \{1\} = g_n(x|q) \quad (45)$$

and

$$g_{n+m}(x|q) := \tilde{\Delta}_x^{n+m} \{1\} = \tilde{\Delta}_x^n g_m(x|q). \quad (46)$$

### 3. Proof of Theorem 1.4

Here, we will give the proof of Theorem 1.4 by using the method of  $q$ -operational equation for Carlitz's  $q$ -operators.

Upon using equation (43), we get:

$$\text{Exp}_q(t\tilde{\Delta}_x\tilde{\Delta}_y)f(x, y) = \text{Exp}_q(xt\tilde{\Delta}_y) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^{\binom{n}{2}+k(k-n)} y^{n-k} t^n}{(q; q)_k (q; q)_{n-k}} f(q^{-n}x, q^{-k}y). \quad (49)$$

Setting  $n \rightarrow n - k$  in equation (49), we have the following:

$$\begin{aligned} & \text{Exp}_q(t\tilde{\Delta}_x\tilde{\Delta}_y)f(x, y) \\ &= \text{Exp}_q(xt\tilde{\Delta}_y) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^{\binom{n+k}{2}-nk} y^n t^{n+k}}{(q; q)_k (q; q)_n} f(q^{-n-k}x, q^{-k}y) \\ &= \text{Exp}_q(xt\tilde{\Delta}_y) \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} t^k}{(q; q)_k} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} y^n t^n}{(q; q)_n} f(q^{-n-k}x, q^{-k}y) \\ &= \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} t^k}{(q; q)_k} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} \text{Exp}_q(xt\tilde{\Delta}_y) \{y^n f(q^{-n-k}x, q^{-k}y)\} \\ &= (-xyt; q)_{\infty} \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} t^k}{(q; q)_k} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{q^{\binom{m}{2}} q^{-nm} y^n (xt)^m}{(q; q)_m} f(q^{-n-k}x, q^{-m-k}y) \\ &= (-xyt; q)_{\infty} \sum_{n,m,k=0}^{\infty} \frac{q^{\binom{n}{2}+\binom{m}{2}+\binom{k}{2}-nm} x^m y^n t^{n+m+k}}{(q; q)_n (q; q)_m (q; q)_k} f(q^{-n-k}x, q^{-m-k}y), \end{aligned}$$

which achieves the proof of Theorem 1.4.

*Proof of Corollary 1.1.* Setting  $f(x, y) = (ax, by; q)_{\infty}$  and  $t = -t$  in Theorem 1.4, we have:

$$\begin{aligned} \text{Exp}_q(-t\tilde{\Delta}_x\tilde{\Delta}_y)(ax, by; q)_{\infty} &= (xyt; q)_{\infty} \sum_{n,m,k=0}^{\infty} \frac{(-1)^{n+m+k} q^{\binom{n}{2}+\binom{m}{2}+\binom{k}{2}-nm} x^m y^n t^{n+m+k}}{(q; q)_n (q; q)_m (q; q)_k} (axq^{-n-k}, byq^{-m-k}; q)_{\infty} \\ &= (xyt; q)_{\infty} \sum_{k,m=0}^{\infty} \frac{q^{\binom{k}{2}+\binom{m}{2}} (byq^{-m-k}; q)_{\infty} (-xt)^m (-t)^k}{(q; q)_k (q; q)_m} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (-ytq^{-m})^n}{(q; q)_n} (axq^{-n-k}; q)_{\infty} \\ &= (xyt; q)_{\infty} \sum_{k,m=0}^{\infty} \frac{q^{\binom{k}{2}+\binom{m}{2}} (byq^{-m-k}; q)_{\infty} (-xt)^m (-t)^k}{(q; q)_k (q; q)_m} (axq^{-k}; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (-ytq^{-m})^n}{(q; q)_n} (axq^{-n-k}; q)_n. \end{aligned}$$

Using (1), we get:

$$\begin{aligned} & \text{Exp}_q(-t\tilde{\Delta}_x\tilde{\Delta}_y)(ax, by; q)_{\infty} \\ &= (xyt, by; q)_{\infty} \sum_{k,m=0}^{\infty} \frac{(q/by; q)_{k+m} (bxytq^{-1})^m (bytq^{-1})^k q^{mk}}{(q; q)_k (q; q)_m} (axq^{-k}; q)_{\infty} \sum_{n=0}^{\infty} \frac{(q^{1+k}/ax; q)_n (axytq^{-1-k-m})^n}{(q; q)_n} \\ &= (xyt, ax, by; q)_{\infty} \sum_{k,m=0}^{\infty} \frac{(q/by; q)_{k+m} (bxytq^{-1})^m (bytq^{-1})^k}{(q; q)_k (q; q)_m} q^{-mk} \frac{(ytq^{-m}; q)_{\infty} (axq^{-k}; q)_k}{(axytq^{-1-k-m}; q)_{\infty}} \\ &= \frac{(xyt, ax, by; q)_{\infty}}{(axyt/q; q)_{\infty}} \sum_{k,m=0}^{\infty} \frac{(q/ax; q)_k}{(q; q)_k} \frac{(q/yt; q)_m (yta^{-1})^m}{(q; q)_m} \frac{(q/by; q)_{m+k} b^{m+k}}{(q^2/axyt; q)_{m+k}}. \end{aligned}$$

This achieves the proof of Corollary 1.1.

## 4. Mehler's Formula for Rogers–Szegő Polynomials

Carlitz [8] established a proof of  $q$ -Mehler's formula for Rogers-Szegő polynomials  $g_n(x|q)$  by using the recursion relation satisfies  $g_n(x|q)$ .

In this section, we will give another proof of  $q$ -Mehler's formula for Rogers-Szegő polynomials given in Theorem 4.1 by using



the  $q$ -exponential operational identity.

*Theorem 4.1 ([8]).* For  $|xyt^2/q| < 1$ , we have:

$$\sum_{n=0}^{\infty} g_n(x|q)g_n(y|q) \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} = \frac{(t, xt, yt, xyt; q)_{\infty}}{(xyt^2/q; q)_{\infty}}. \quad (50)$$

*Proof of Theorem 4.1.* Replace  $t$  by  $t\tilde{\Delta}_x$  in the generating function (14), we get:

$$\sum_{n=0}^{\infty} g_n(y|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} \tilde{\Delta}_x^n = \text{Exp}_q(-t\tilde{\Delta}_x) \text{Exp}_q(-ty\tilde{\Delta}_x). \quad (51)$$

For  $f(x) \equiv 1$  in both sides of (51), we obtain:

$$\sum_{n=0}^{\infty} g_n(y|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} \tilde{\Delta}_x^n \cdot \{1\} = \text{Exp}_q(-t\tilde{\Delta}_x) \text{Exp}_q(-ty\tilde{\Delta}_x) \cdot \{1\}. \quad (52)$$

Using the identity (45) on the left-hand side and (32) in the right-hand side of (52), respectively, we gain:

$$\sum_{n=0}^{\infty} g_n(y|q)g_n(x|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} = \frac{(t, xt, yt, xyt; q)_{\infty}}{(xyt^2/q; q)_{\infty}},$$

which is the desired result.

Next, comparing (25) and (52), one obtains:

$$\text{Exp}_q(-t\tilde{\Delta}_x\tilde{\Delta}_y) \cdot \{1\} = \text{Exp}_q(-t\tilde{\Delta}_x) \text{Exp}_q(-ty\tilde{\Delta}_x) \cdot \{1\}.$$

In what follows, we will give another proof for extended Mehler's formula for the Rogers-Szegő polynomial stated in the above Theorem.

*Theorem 4.2 ([5]).* For  $|xyt^2/q| < 1$ , we have:

$$\begin{aligned} & \sum_{n=0}^{\infty} g_{n+m}(x|q)g_n(y|q) \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} \\ &= \frac{x^m (t, xt, yt, xyt; q)_{\infty}}{(xyt^2/q; q)_{\infty}} {}_3\Phi_2 \left[ \begin{matrix} q^{-m}, \frac{q}{xt}, \frac{q}{xyt}; \\ 0, \frac{q^2}{xyt^2}; \end{matrix} q; q \right], \end{aligned} \quad (53)$$

$$= \frac{(t, xt, yt, xyt; q)_{\infty}}{(xyt^2/q; q)_{\infty}} \frac{(q/t; q)_m}{(ty/q)^m (q^2/(xyt^2); q)_m} {}_2\Phi_1 \left[ \begin{matrix} q^{-m}, \frac{q}{xyt}; \\ tq^{-m}; \end{matrix} q; ty \right]. \quad (54)$$

*Proof of Theorem 4.2.* Using the identity (45), we find that the left-hand side of (53) is equivalent to:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(-t)^n q^{\binom{n}{2}}}{(q; q)_n} \tilde{\Delta}_x^{n+m} \tilde{\Delta}_y^n \cdot \{1\} \\ &= \tilde{\Delta}_x^m \cdot \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} (-t\tilde{\Delta}_x\tilde{\Delta}_y)^n \cdot \{1\} \\ &= \tilde{\Delta}_x^m \text{Exp}_q(-t\tilde{\Delta}_x\tilde{\Delta}_y) \cdot \{1\}. \end{aligned} \quad (55)$$

Making used of Corollary 1.2, the equation (55) reads:

$$\tilde{\Delta}_x^m \cdot \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} (-t\tilde{\Delta}_x\tilde{\Delta}_y)^n \cdot \{1\} = \tilde{\Delta}_x^m \frac{(t, xt, yt, xyt; q)_{\infty}}{(xyt^2/q; q)_{\infty}}. \quad (56)$$

Next, using (43), we obtain:

$$\tilde{\Delta}_x^m \frac{(xt, xyt; q)_\infty}{(xyt^2/q; q)_\infty} = \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q q^{k(k-m)} x^{m-k} \frac{(xtq^{-k}, xytq^{-k}; q)_\infty}{(xyt^2q^{-k}/q; q)_\infty}. \quad (57)$$

From the relations (1) and (12), respectively, we gain:

$$\begin{aligned} \tilde{\Delta}_x^m \frac{(t, xt, yt, xyt; q)_\infty}{(xyt^2/q; q)_\infty} &= \frac{x^m(t, xt, yt, xyt; q)_\infty}{(xyt^2/q; q)_\infty} \sum_{k=0}^m \frac{(q^{-m}, q/(xt), q/(xyt); q)_k q^k}{(q, q^2/xyt^2; q)_k} \\ &= \frac{x^m(t, xt, yt, xyt; q)_\infty}{(xyt^2/q; q)_\infty} {}_3\Phi_2 \left[ \begin{matrix} q^{-m}, \frac{q}{xt}, \frac{q}{xyt}; \\ 0, \frac{q^2}{xyt^2}; \end{matrix} q; q \right]. \end{aligned} \quad (58)$$

Setting  $a = q/xyt, b = 0, c = q/xt$  and  $d = q^2/xyt^2$  in the  ${}_3\Phi_2$  transformation [14]

$${}_3\Phi_2 \left[ \begin{matrix} q^{-m}, a, c; \\ b, d; \end{matrix} q; q \right] = \frac{a^m(d/a; q)_m}{(d; q)_m} {}_3\Phi_2 \left[ \begin{matrix} q^{-m}, a, \frac{b}{c}; \\ b, \frac{aq^{1-m}}{d}; \end{matrix} q; \frac{cq}{d} \right], \quad (59)$$

and substituting the resulting relation into (58), we achieve the proof of Theorem 4.2.

*Theorem 4.3.* For  $|xyt^2q| < 1$  and  $n, m \in \mathbb{N}$ , we have:

$$\begin{aligned} \sum_{k=0}^{\infty} g_{n+k}(x|q)g_{m+k}(y|q) \frac{(-1)^k q^{\binom{k}{2}} t^k}{(q; q)_k} \\ = \frac{x^n y^m (t, xt, yt, xyt; q)_\infty}{(xyt^2/q; q)_\infty} \sum_{k=0}^n \frac{(q^{-n}, q/xt, q/xyt; q)_k q^k}{(q, q^2/xyt^2; q)_k} {}_3\Phi_2 \left[ \begin{matrix} q^{-m}, \frac{q}{yt}, \frac{q^{1+k}}{xyt}; \\ 0, \frac{q^{2+k}}{xyt^2}; \end{matrix} q; q \right]. \end{aligned} \quad (60)$$

*Remark 4.1.* Taking  $m = 0$ , Theorem 4.3 reduces to the Cao's concluding remarks [5] given by Theorem 4.2. Setting  $(n, m) = (0, 0)$ , Theorem 4.3 reduces to the Carlitz's [8] concluding remarks given by Theorem 4.1.

*Proof of Theorem 4.3.* Upon using (46), we have:

$$\begin{aligned} \sum_{k=0}^{\infty} g_{n+k}(x|q)g_{m+k}(y|q) \frac{(-1)^k q^{\binom{k}{2}} t^k}{(q; q)_k} &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} t^k}{(q; q)_k} \tilde{\Delta}_x^{n+k} \tilde{\Delta}_y^{m+k} \\ &= \tilde{\Delta}_x^n \tilde{\Delta}_y^m \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_k} (t \tilde{\Delta}_x \tilde{\Delta}_y)^k \\ &= \tilde{\Delta}_x^n \tilde{\Delta}_y^m \text{Exp}_q \left( -t \tilde{\Delta}_x \tilde{\Delta}_y \right) \cdot \{1\}. \end{aligned} \quad (61)$$

Applying Proposition 1.2, the right-hand side of (61) reads:

$$\tilde{\Delta}_x^n \tilde{\Delta}_y^m \text{Exp}_q \left( -t \tilde{\Delta}_x \tilde{\Delta}_y \right) \cdot \{1\} = (t; q)_\infty \tilde{\Delta}_x^n \tilde{\Delta}_y^m \left\{ \frac{(xt, yt, xyt; q)_\infty}{(xyt^2/q; q)_\infty} \right\}. \quad (62)$$

Applying (43) to the right-hand side of (62), we get:

$$\begin{aligned} &\tilde{\Delta}_x^n \tilde{\Delta}_y^m \text{Exp}_q \left( -t \tilde{\Delta}_x \tilde{\Delta}_y \right) \cdot \{1\} \\ &= (t; q)_\infty \tilde{\Delta}_x^n \tilde{\Delta}_y^m \left\{ \frac{(xt, yt, xyt; q)_\infty}{(xyt^2/q; q)_\infty} \right\} \\ &= \sum_{k=0}^n \sum_{j=0}^m \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ j \end{bmatrix}_q q^{k(k-n)+j(j-m)} x^{n-k} y^{m-j} \frac{(t, xtq^{-k}, yqtq^{-j}, xytq^{-k-j}; q)_\infty}{(xyt^2q^{-1-k-j}; q)_\infty}. \end{aligned} \quad (63)$$

By (1), we gain:

$$\begin{aligned}
 & \tilde{\Delta}_x^n \tilde{\Delta}_m^m \text{Exp}_q \left( -t \tilde{\Delta}_x \tilde{\Delta}_y \right) \cdot \{1\} \\
 &= \frac{(t, xt, yt, xyt; q)_\infty}{(xyt^2/q; q)_\infty} \sum_{k=0}^n \sum_{j=0}^m \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ j \end{bmatrix}_q q^{k(k-n)+j(j-m)} x^{n-k} y^{m-j} \frac{(xtq^{-k}; q)_k (ytq^{-j}; q)_j (xytq^{-k-j}; q)_{k+j}}{(xyt^2 q^{-1-k-j}; q)_{k+j}} \\
 &= \frac{x^n y^m (t, xt, yt, xyt; q)_\infty}{(xyt^2/q; q)_\infty} \sum_{k=0}^n \sum_{j=0}^m \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ j \end{bmatrix}_q q^{\binom{k+1}{2} + \binom{j+1}{2} - nk - mj} (-1)^k (-1)^j \frac{(q/xt; q)_k (q/yt; q)_j (q/xyt; q)_{k+j}}{(q^2/xyt^2; q)_{k+j}} \\
 &= \frac{x^n y^m (t, xt, ytq, xyq; q)_\infty}{(xyt^2/q; q)_\infty} \sum_{k=0}^n \frac{(q^{-n}, q/xt, q/xyt; q)_k q^k}{(q, q^2/xyt^2; q)_k} \sum_{j=0}^m \frac{(q^{-m}, q/yt, q^{1+k}/xyt; q)_j q^j}{(q, q^{2+k}/xyt^2; q)_j} \\
 &= \frac{x^n y^m (t, xt, yt, xyt; q)_\infty}{(xyt^2/q; q)_\infty} \sum_{k=0}^n \frac{(q^{-n}, q/xt, q/xyt; q)_k q^k}{(q, q^2/xyt^2; q)_k} {}_3\Phi_2 \left[ \begin{matrix} q^{-m}, \frac{q}{yt}, \frac{q^{1+k}}{xyt}; \\ 0, \frac{q^{2+k}}{xyt^2}; \end{matrix} q; q \right]. \tag{64}
 \end{aligned}$$

We get the desire result, by summarizing the above calculations (61) to (64) and making used relation (12). The proof of Theorem 4.3 is complete.

Upon setting  $x = -1$  in (14) and using the fact that  $(-t, t; q)_\infty = (t^2; q^2)_\infty$ , we obtain:

$$\sum_{n=0}^{\infty} g_n(-1|q) \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} = (t^2; q^2)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2\binom{n}{2}} t^{2n}}{(q^2; q^2)_n}. \tag{65}$$

Equating the coefficients of powers of  $t^n$  on both sides of (65), we obtain:

$$g_{2n+1}(-1|q) = 0 \tag{66}$$

and

$$g_{2n}(-1|q) = (-1)^n q^{n^2 - n - \binom{2n}{2}} \frac{(q; q)_{2n}}{(q^2; q^2)_n} = (-1)^n q^{-n^2} (q; q^2)_n. \tag{67}$$

Next, taking  $m = 0$  and  $y = -1$  in (60) and using the fact (66) and (67) are satisfied, we get:

$$\sum_{k=0}^{\infty} g_{n+2k}(x|q) g_{2k}(-1|q) \frac{q^{\binom{2k}{2}} t^{2k}}{(q; q)_{2k}} = \frac{x^n (t^2, x^2 t^2; q^2)_\infty}{(-xt^2/q; q)_\infty} \sum_{k=0}^n \frac{(q^{-n}; q)_k (q^2/x^2 t^2; q^2)_k q^k}{(-q^2/xt^2, q; q)_k}.$$

Setting  $t^2$  by  $t$ , we obtain that for  $|xt/q| < 1$ ,

$$\sum_{k=0}^{\infty} g_{n+2k}(x|q) \frac{q^{k^2 - k} t^k}{(q^2; q^2)_k} = \frac{x^n (t, tx^2; q^2)_\infty}{(-xt/q; q)_\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2} - nk} \frac{(q^2/tx^2; q^2)_k}{(-q^2/tx; q)_k}, \tag{68}$$

which is a  $q^{-1}$ -version of Doetsch's formula [21].

*Proposition 4.1.* For  $n \in \mathbb{N}$ , we have the following  $q$ -exponential operator identity:

$$\text{Exp}_q(-tq\tilde{\Delta}_x) g_n(x|q) = (tq, xtq; q)_\infty \psi_n^{(t)}(x|q), \tag{69}$$

where [1]

$$\psi_n^{(a)}(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k (aq^{1-k}; q)_k. \tag{70}$$

*Proof of Proposition 4.1.* For  $f(x) = g_n(x|q)$ , the relation (20) reads:

$$\text{Exp}_q(-tq\tilde{\Delta}_x) g_n(x|q) = \sum_{k=0}^{\infty} \frac{(-tq)^k q^{\binom{k}{2}}}{(q; q)_k} \tilde{\Delta}_x^k g_n(x|q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} t^k}{(q; q)_k} g_{n+k}(x|q).$$

On the other hand, taking  $f(x) = g_n(x|q)$ , the relation (20) reads:

$$\text{Exp}_q(-tq\tilde{\Delta}_x)g_n(x|q) = (xtq; q) \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} t^n}{(q; q)_k} g_n(q^{-k}x|q). \quad (71)$$

Using (11), we obtain:

$$\text{Exp}_q(-tq\tilde{\Delta}_x)g_n(x|q) = (xtq; q) \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}} t^k}{(q; q)_k} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-n)} q^{-kj} x^j. \quad (72)$$

Interchanging the order of summation in the right-hand side of (72), we get:

$$\begin{aligned} \text{Exp}_q(-tq\tilde{\Delta}_x)g_n(x|q) &= (xtq; q) \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-n)} x^j \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (tq^{1-j})^k}{(q; q)_k} \\ &= (xtq; q) \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-n)} x^j (tq^{1-j}; q)_{\infty} \\ &= (tq, xtq; q)_{\infty} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-n)} (tq^{1-j}; q)_j x^j \\ &= (tq, xtq; q)_{\infty} \psi_n^{(t)}(x|q), \end{aligned}$$

which is the right-hand side of (69). This achieves the proof of Proposition 4.1.

## 5. $q$ -Nielsen's Formulas for Rogers-Szegö Polynomials

In this section, we will give  $q$ -Burchall formula for Rogers-Szegö polynomials  $g_n(x|q)$  as an application.

*Proposition 5.1.* For any non-negative integers  $m$  and  $n$ , each of the following  $q$ -Burchall formula for Rogers-Szegö polynomials  $g_n(x|q)$  holds true:

$$g_{n+m}(x|q) = \sum_{k=0}^{\min(n,m)} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q (q; q)_k x^k q^{k(k-n-m)} g_{n-k}(x|q) g_{m-k}(x|q) \quad (73)$$

and

$$g_n(x|q) g_m(x|q) = \sum_{k=0}^{\min(n,m)} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q (q; q)_k (-x)^k q^{\binom{k}{2} + k(k-n-m)} g_{n+m-2k}(x|q). \quad (74)$$

Upon putting  $m = 0$  in Proposition 5.1, we conclude that:

$$g_n(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (q; q)_k q^{k(k-n)} x^k g_{n-k}(x|q), \quad (75)$$

and

$$g_n(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (q; q)_k q^{\binom{k}{2} + k(k-n)} (-x)^k g_{n-2k}(x|q). \quad (76)$$

*Proof of Proposition 5.1.* Taking  $f(x) = g_m(x|q)$  in (22), we get:

$$\tilde{\Delta}_x^n g_m(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2} - k(n+1)} x^k g_{n-k}(x|q) \theta_x^k g_m(x|q). \quad (77)$$

Using the fact that

$$\tilde{\Delta}_x^n g_m(x|q) = g_{n+m}(x|q)$$

and

$$\theta_x^k g_m(x|q) = q^{\binom{k+1}{2} - mk} \frac{(q; q)_m}{(q; q)_{m-k}} g_{m-k}(x|q), \quad k \leq m,$$

we immediately obtain:

$$g_{n+m}(x|q) = \sum_{k=0}^{\min(n,m)} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q (q; q)_k x^k q^{k(k-n-m)} g_{n-k}(x|q) g_{m-k}(x|q). \quad (78)$$

This achieves the proof of Proposition 5.1.

Before proving (74), let us turn back to (17).

$$\sum_{n,m=0}^{\infty} g_{n+m}(x|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} = \frac{(s, t, sx, tx; q)_{\infty}}{(stx/q; q)_{\infty}}. \quad (79)$$

Using the generating function of  $g_n(x|q)$ , we have:

$$\sum_{n,m=0}^{\infty} g_n(x|q) g_m(x|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} = (s, t, sx, tx; q)_{\infty}. \quad (80)$$

Divided by  $(stx/q; q)_{\infty}$  both sides of (80) and comparing (79), we obtain:

$$\begin{aligned} & \frac{1}{(stx/q; q)_{\infty}} \sum_{n,m=0}^{\infty} g_n(x|q) g_m(x|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \\ &= \sum_{n,m=0}^{\infty} g_{n+m}(x|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \end{aligned}$$

or equivalent to

$$\begin{aligned} & \sum_{n,m=0}^{\infty} g_n(x|q) g_m(x|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \\ &= (stx/q; q)_{\infty} \sum_{n,m=0}^{\infty} g_{n+m}(x|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m}. \end{aligned} \quad (81)$$

In view of (5), the right-hand side of (81) reads

$$\begin{aligned} & \sum_{n,m=0}^{\infty} g_{n+m}(x|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} (-sxtq^{-1})^k}{(q; q)_k} \\ &= \sum_{n,m,k=0}^{\infty} g_{n+m}(x|q) (-1)^{n+m+k} q^{\binom{n}{2} + \binom{m}{2} + \binom{k}{2}} \frac{s^{n+k}}{(q; q)_n} \frac{t^{m+k}}{(q; q)_m} \frac{(xq^{-1})^k}{(q; q)_k}. \end{aligned} \quad (82)$$

Combining (81) and (82), we obtain:

$$\begin{aligned} & \sum_{n,m=0}^{\infty} g_n(x|q) g_m(x|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \\ &= \sum_{n,m,k=0}^{\infty} g_{n+m}(x|q) (-1)^{n+m+k} q^{\binom{n}{2} + \binom{m}{2} + \binom{k}{2}} \frac{s^{n+k}}{(q; q)_n} \frac{t^{m+k}}{(q; q)_m} \frac{(xq^{-1})^k}{(q; q)_k}. \end{aligned}$$

Equating the coefficients of  $s^n t^m$  in both sides of the above equation, we achieve the proof of Proposition 5.1.

*Theorem 5.1.* Each of the following formula for Rogers-Szegő polynomials  $g_n(x|q)$  holds true:

$$\begin{aligned} & \sum_{2k \leq n} \frac{(q; q)_n (xyq^{-1})^k}{(q; q)_{n-2k} (q; q)_k} (-1)^k q^{\binom{k+1}{2} + 2k(k-n)} g_{n-2k}(x|q) g_{n-2k}(y|q) \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k g_k(y|q) g_{n-k}(y|q), \end{aligned} \quad (83)$$

$$= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k g_{n-k} \left( \frac{y}{x} | q \right) g_k(xy|q). \quad (84)$$

*Proof of Theorem 5.1.* Using the generating function of  $g_n(x|q)$ , we have:

$$\sum_{n,m=0}^{\infty} g_n(y|q) g_m(y|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{t^n}{(q; q)_n} \frac{(xt)^m}{(q; q)_m} = (t, yt, xt, xyt; q)_{\infty}. \quad (85)$$

Next, using the Mehler's formula (50), we have

$$\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} g_n(x|q) g_n(y|q) \frac{t^n}{(q; q)_n} = \frac{(t, xt, yt, xyt; q)_{\infty}}{(xyt^2/q; q)_{\infty}}. \quad (86)$$

Comparing (85) and (86), we obtain:

$$\sum_{n,m=0}^{\infty} g_n(y|q) g_m(y|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{t^n}{(q; q)_n} \frac{(xt)^m}{(q; q)_m} = (xyt^2/q; q)_{\infty} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} g_n(x|q) g_n(y|q) \frac{t^n}{(q; q)_n},$$

which equals to:

$$\sum_{n,k=0}^{\infty} g_n(y|q) g_k(y|q) (-1)^{n+k} q^{\binom{n}{2} + \binom{k}{2}} \frac{x^k t^{n+k}}{(q; q)_n (q; q)_k} = \sum_{n,k=0}^{\infty} (-1)^{n+k} q^{\binom{n}{2} + \binom{k}{2}} g_n(x|q) g_n(y|q) \frac{(xyq^{-1})^k t^{n+2k}}{(q; q)_n (q; q)_k}.$$

Interchanging the order of summation in both sides of the above equation, we obtain:

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^n g_{n-k}(y|q) g_k(y|q) q^{\binom{n-k}{2} + \binom{k}{2}} \frac{x^k t^n}{(q; q)_{n-k} (q; q)_k} \\ &= \sum_{n,k=0}^{\infty} (-1)^k q^{\binom{n-2k}{2} + \binom{k}{2}} g_{n-2k}(x|q) g_{n-2k}(y|q) \frac{(xyq^{-1})^k t^n}{(q; q)_{n-2k} (q; q)_k}. \end{aligned} \quad (87)$$

Equating the coefficients of  $t^n$  in both sides of (87), we gain:

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k g_{n-k}(y|q) g_k(y|q) = \sum_{2k \leq n} \frac{(q; q)_n (xyq^{-1})^k}{(q; q)_{n-2k} (q; q)_k} (-1)^k q^{\binom{k+1}{2} + 2k(k-n)} g_{n-2k}(x|q) g_{n-2k}(y|q),$$

which is the right-hand side of (5.1).

Similarly, noting the fact that

$$\sum_{m,n=0}^{\infty} g_n \left( \frac{y}{x} | q \right) g_n(xy|q) (-1)^{n+m} q^{\binom{n}{2} + \binom{m}{2}} \frac{(xt)^n t^m}{(q; q)_n (q; q)_m} = (t, xt, yt, xyt; q)_{\infty}$$

and using the same calculations as above, we get the desire result. This achieves the proof of Theorem 5.1.

## 6. Mehler Formula for the Hahn Polynomials

Al-Salam and Carlitz [1] found the following bilinear generating function by the transformation theory of  $q$ -series. Here, we will give a simply proof of the Mehler formula for the Hahn polynomials  $\psi_k^{(a)}(x|q)$  by the  $q$ -exponential operational identity.

*Theorem 6.1.* We have:

$$\begin{aligned} & \sum_{k=0}^{\infty} \psi_k^{(a)}(x|q) \psi_k^{(b)}(y|q) \frac{(-1)^k q^{\binom{k}{2}} z^k}{(q; q)_k} \\ &= \frac{(q^2 abx, qbx; q)_{\infty}}{(abxq; q)_{\infty}} \sum_{n,m=0}^{\infty} \frac{(1/ax; q)_n}{(q; q)_n} \frac{(1/bz; q)_m}{(q; q)_m} \frac{(qbxz^{-1})^m}{(1/abxq; q)_{n+m}} \frac{(1/by; q)_{n+m} y^{n+m}}{(q^2/abxq; q)_{n+m}}, \end{aligned} \quad (88)$$

provided  $|abxq| < 1$ .

Before proving Theorem 6.1, we need the following:

*Proposition 6.1.* We have:

$$\begin{aligned} & \sum_{m,n,k=0}^{\infty} q^{\binom{m}{2} + \binom{n}{2} + \binom{k}{2}} g_{m+k}(a|q) g_{n+k}(b|q) \frac{(-x)^m (-y)^n (-z)^k}{(q; q)_m (q; q)_n (q; q)_k} \\ &= \frac{(x, y, xyz, ax, by, yz; q)_{\infty}}{(axyz/q; q)_{\infty}} \sum_{n,m=0}^{\infty} \frac{(q/ax; q)_n}{(q; q)_n} \frac{(q/yz; q)_m}{(q; q)_m} \frac{(yza^{-1})^m}{(q^2/axyz; q)_{n+m}} \frac{(q/by; q)_{n+m} b^{n+m}}{(q^2/axyz; q)_{n+m}}, \end{aligned} \quad (89)$$

provided that  $|axyz/q| < 1$ .

*Remark 6.1.* Setting  $y = b = 0$  in Proposition 6.1, we get Carlitz's [8] the concluding remarks.

*Proof of Proposition 6.1.* Using the identity (45), we find that the left-hand side of (89) is equivalent to:

$$\begin{aligned} & \sum_{m,n,k=0}^{\infty} (-1)^{n+m+k} q^{\binom{m}{2} + \binom{n}{2} + \binom{k}{2}} \frac{x^m y^n z^k}{(q; q)_m (q; q)_n (q; q)_k} \tilde{\Delta}_a^{m+k} \tilde{\Delta}_b^{n+k} \cdot \{1\} \\ &= \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{(q; q)_k} (-z \tilde{\Delta}_a \tilde{\Delta}_b)^k \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} (-y \tilde{\Delta}_b)^n \sum_{m=0}^{\infty} \frac{q^{\binom{m}{2}}}{(q; q)_m} (-x \tilde{\Delta}_a)^m \cdot \{1\} \\ &= \text{Exp}_q(-z \tilde{\Delta}_b \tilde{\Delta}_a) \text{Exp}_q(-y \tilde{\Delta}_b) \text{Exp}_q(-x \tilde{\Delta}_a) \cdot \{1\}. \end{aligned} \quad (90)$$

Next, making used corollary 2.1, the equations

$$\begin{aligned} \text{Exp}_q(-x \tilde{\Delta}_a) \cdot \{1\} &= (x, ax; q)_{\infty}, \\ \text{Exp}_q(-y \tilde{\Delta}_b) \cdot \{1\} &= (y, by; q)_{\infty} \end{aligned} \quad (91)$$

and Proposition 1.1, the right-hand side of (90) reads:

$$\begin{aligned} & (x, y; q)_{\infty} \text{Exp}_q(-z \tilde{\Delta}_x \tilde{\Delta}_y) (ax, by; q)_{\infty} \\ &= \frac{(x, y, xyz, ax, by, yz; q)_{\infty}}{(axyz/q; q)_{\infty}} \sum_{n,m=0}^{\infty} \frac{(q/ax; q)_n}{(q; q)_n} \frac{(q/yz; q)_m}{(q; q)_m} \frac{(yza^{-1})^m}{(q^2/axyz; q)_{n+m}} \frac{(q/by; q)_{n+m} b^{n+m}}{(q^2/axyz; q)_{n+m}}, \end{aligned}$$

which is the right-hand side of (89). This achieves the proof of Proposition 6.1.

*Proof of Theorem 6.1.* Making use of Proposition 4.1, we have:

$$\text{Exp}_q(-aq \tilde{\Delta}_x) g_k(x|q) = (aq, axq; q)_{\infty} \psi_k^{(a)}(x|q),$$

and

$$\text{Exp}_q(-bq \tilde{\Delta}_y) g_k(y|q) = (bq, byq; q)_{\infty} \psi_k^{(b)}(y|q).$$

Immediately, the right-hand side of (88) reads:

$$\frac{1}{(aq, bq, axq, byq; q)_\infty} \sum_{k=0}^{\infty} \text{Exp}_q(-bq\tilde{\Delta}_y)g_k(y|q) \text{Exp}_q(-aq\tilde{\Delta}_x)g_k(x|q) \frac{(-1)^k q^{\binom{k}{2}} z^k}{(q; q)_k}. \quad (92)$$

By using (71), it becomes:

$$\begin{aligned} & \frac{1}{(aq, bq, axq, byq; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} z^k}{(q; q)_k} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (qa)^n}{(q; q)_n} g_{n+k}(x|q) \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (qb)^m}{(q; q)_m} g_{m+k}(y|q) \\ &= \frac{1}{(aq, bq, axq, byq; q)_\infty} \sum_{k=0}^{\infty} (-1)^{k+n+m} q^{\binom{k}{2} + \binom{n}{2} + \binom{m}{2}} g_{n+k}(x|q) g_{m+k}(y|q) \frac{(qa)^n (qb)^m z^k}{(q; q)_k (q; q)_n (q; q)_m}. \end{aligned}$$

Setting  $(a, b, x, y) = (x, y, aq, bq)$  in Proposition 6.1, the right-hand side of the above expression becomes:

$$\frac{(q^2 abz, qbz; q)_\infty}{(abxzq; q)_\infty} \sum_{n,m=0}^{\infty} \frac{(1/ax; q)_n}{(q; q)_n} \frac{(1/bz; q)_m (qbxz^{-1})^m}{(q; q)_m} \frac{(1/by; q)_{n+m} y^{n+m}}{(1/abxz; q)_{n+m}}.$$

The proof of Theorem 6.1 is complete.

## 7. Conclusion

Based on methods of Carlitz's  $q$ -operators and  $q$ -operational equation presented respectively [4, 9, 22], we constructed a  $q$ -operational equation for Carlitz's  $q$ -operators and gave some applications to solve several kinds of generating functions for Rogers–Szegő polynomials and Hahn polynomials. We showed through several generating functions examples that the theory can be applied, which gives us quickly the results. One of the main features of this method is that it provides an effective approach to calculate problems of generating functions for  $q$ -polynomials.

This paper is a further extension of Cao [4] results. We can use this method to calculate all generating functions for Rogers–Szegő polynomials [10] and Hahn polynomials [25]. We believe that using this method can prove more generating functions for  $q$ -polynomials. Since we have already provided some results, we have omitted the details of generating functions for other  $q$ -polynomials.

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