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Adaptive Type-II Hybrid Progressive Schemes Based on Maximum Product of Spacings for Parameter Estimation of Kumaraswamy Distribution

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Abstract: The present article aims to illustrate how the Adaptive Type-II Progressive Hybrid censoring scheme can be used to make statistical inferences regarding the shape parameters of the Kumaraswamy distribution. By adopting this scheme, one can reduce the total testing time and the cost associated with the failure of the units. Best of all, one can increase the effectiveness of the statistical analysis while reducing the total test time. The maximum product of spacings method (MPS) in classical estimation settings is highly effective. According to several authors, this method is a superior alternative to the maximum likelihood estimation method (MLE), which delivers more accurate estimates than the maximum likelihood estimation method. Our goal in this article is to estimate the shape parameters of the Kumaraswamy distribution by utilizing the MPS method. Asymptotic normality properties of the estimators are implemented to obtain approximate confidence intervals. In addition, bootstrap confidence intervals are calculated. Monte Carlo simulations have been carried out to compare the MPS and MLE methods. In order to assess the effectiveness of the proposed procedure, a numerical example based on real data is presented.

Keywords: Component, Formatting, Style, Styling, Insert

1. Introduction

In survival analysis, a life test is conducted on a sample of size n to observe their failure times. A time-to-failure model is then developed based on the data collected during the test. Nevertheless, such a strategy would be expensive, time-consuming, and impractical. Furthermore, time constraints and facilities limitations may require the experimenter to halt the study before recording the failure times for all subjects. In addition, it may be necessary to exclude some functioning test subjects from the test to gather degradation-related information about failure times or for another research endeavor. It is generally the case when the tested subject is expensive, such as when clinical equipment is needed. Occasionally, failures are deliberate and predictable. In such cases, censored samples are created. Progressive censoring is widely used in life testing studies to address various concerns that experimenters may have, such as reducing total test time, conserving experimental units, and developing efficient estimation methods. Nevertheless, there must be a trade-off between these three concerns to reduce the experiment's cost and time.

There are two main types of censoring: Type-I and Type-II. For Type-I censoring, the experiment is terminated at a predetermined time T, while for Type-II censoring, the experiment is terminated once a specific number of failures, m, has been observed. To further reduce experimental time and cost, a hybrid censoring scheme was used, which is a mixture of Type-I and Type-II censoring schemes. As a result of this scheme, the experiment ends at a predefined time. Nevertheless, none of these censoring schemes allow for intermediate removal of active subjects throughout the test other than at the final termination point.

Due to this inflexibility, various progressive censoring schemes have been developed in the literature to model the removal of subjects during lifetime experiments. A significant number of authors have discussed inference under progressive censoring with different lifetime distributions. In addition to Balakrishnan and Asgharzadeh [3], Kim et al. [19], Helu and Samawi [15], Helu et al. [16] and recently, Nassar et al. [25]. A comprehensive analysis of progressive censoring is provided by Balakrishnan and Cramer [4].

In Progressive Type-II censoring, n independent items are placed at the same time on a life testing experiment and only $m(< n)$ failures are completely observed. The censoring occurs progressively in m stages as follows, when the first failure is observed, a random sample of size R_1 is immediately drawn and removed from the test. Then after the failure of the second item, another sample of size R_2 is randomly selected and removed from the remaining survival units. Continuing this process until the *mth* failure, $X_{m:m:n}$, is observed and all remaining $R_m = n - R - \cdots - R_{m-1} - m$ surviving units are removed from the experiment, with $X_{1:m:n} \leq X_{2:m:n} \leq \cdots \leq X_{m:m:n}$ being the ordered failure times resulting from the Progressively Type-II censored experiment. For notation simplicity, we will write X_i for $X_{i:m:n}$.

One major drawback of the Progressive Type-II censoring scheme is that the experiment may take a long time if the subjects are highly reliable. Kundu and Joarder [22] proposed the Progressive Type-II hybrid (P-II hybrid) censoring scheme to address this issue. The P-II hybrid scheme is an amalgamation of the hybrid and the progressive censoring schemes. It is considered to be more flexible according to Kundu and Joarder [22], Panahi [26], and Wang [38]. A P-II hybrid censoring experiment terminates at a predetermined time T^*=$min(T, X_m)$, where $T > 0$ and the integer m are pre-assigned. It is pertinent to note that, in this scheme, the total time required to terminate the experiment does not exceed T. However, the P-II hybrid-censoring scheme has the disadvantage that the number of observed failures is random. As a result, it is possible that it will be a very small number (even zero), which means that traditional statistical inference methods may not be valid or may not be efficient in estimating the model parameter(s). In order to overcome this disadvantage, Ng et al. [24] introduced an adaptation of the P-II hybrid censoring scheme called the Adaptive Type-II Progressive Hybrid Censoring (Adaptive-IIPH). This enhanced scheme, not only saves the total test time and the cost induced by the failure of the units but also increases the efficiency of the statistical analysis. This enhancement also ensures that m items are obtained.

In recent years, the Adaptive-IIPH censoring scheme has been studied by a vast number of authors, among others, we list Cui et al. [8], Ye et al. [40], Zheng and Shi [41], Kohansal and Shoaee [20], Yan and Wang [39] and recently Panahi and Asadi [27]. The maximum likelihood method (MLE) is often regarded as one of the most powerful and acceptable approaches for drawing statistical inferences due to its consistency, sufficiency, invariance, asymptotic efficiency, and, more importantly, ease of calculation. Despite this, Pitman [28], Cheng and Amin [6], and Ranneby [31] demonstrated that the MLE method breaks down due to unboundedness of the likelihood in situations such as mixtures of continuous distributions, heavy-tailed distributions, and J-shaped distributions. Huzurbazar [17] said that when the range depends upon the parameter, the likelihood equation has no consistent solution under certain conditions. Clearly, this demonstrates the likelihood principle's flaw, which leads to estimates approaching the smallest order statistic, resulting in an unbounded likelihood function in the constrained parameter space. Consequently, parameter estimates may be inconsistent; see Harter and Moore [14].

Considering these shortcomings, Cheng and Amin [6] proposed a maximum product of spacings (MPS) estimator that would overcome these problems by returning valid results across a broader range of distributions. Cheng and Amin's method is more intuitive, and some might see it as a pragmatic answer to the challenges associated with the likelihood (Titterington [37]). Ranneby [31] further justified using the MPS estimator by showing that it possesses similar properties as the MLE, including asymptotically sufficient performance, but is more robust for various classes of problems. Shao and Hahn [33], Cheng and Amin [6], and Ghosh and Jammalamadaka [13] showed that under classical setup the MPS method is able to provide estimators that possess most of the large sample optimum properties like sufficiency, consistency, and asymptotic efficiency, which are also being possessed by the MLE. Cheng and Amin [7] used examples to demonstrate the unbiasedness, consistency, and efficiency features of the MPS. Most importantly, the invariance property of the MPS is similar to that of the MLE, as shown by Coolen and Newby [7].

In reliability studies, small sample sizes are prevalent, and the MPS estimators outperform the MLE in this aspect. As a result, the MPS is an excellent method for dependability research (Anatolyev and Kosenok [2]). El-Sherpieny et al. [11] estimated the parameters of the power Lomax distributed using the maximum product of spacing when data are Progressively Type-II hybrid censoring.

Almetwally and Almongy [1] sought to estimate the parameters of the generalized power Weibull distribution under Progressive Type-II censored samples using the maximum product of spacing, the maximum likelihood, and the Bayesian approaches. Coolen and Newby [7] developed a Bayes estimator based on the MPS method, which is compatible with the usual posterior distribution as the MPS is asymptotically equivalent to the likelihood function. Singh et al. [34] proposed a Bayesian model for analyzing a completely observed sample using an exponential distribution. The model was further developed by Singh et al. [35] for a censored sample from a generalized inverted exponential distribution.

Pyke [30] showed that for an ordered sample of size m drawn from a population with cumulative distribution function $F(x, \theta)$ there are $(m + 1)$ first order-spacings, as follows:

$$\begin{aligned} D_1 &= F(x_1, \theta), \\ D_{m+1} &= 1 - F(x_m, \theta), \\ D_i &= F(x_i, \theta) - F(x_{i-1}, \theta)\; i = 1, 2, ..., m, \end{aligned} \tag{1}$$

The Kumaraswamy distribution which is denoted by $Kum(\alpha, \lambda)$ provides a population model which is useful in several areas of statistics, including life testing, reliability and hydrological studies. The probability density function (*pdf*), and the cumulative distribution function (*cdf*) of $Kum(\alpha, \lambda)$ are given, respectively, by

$$f(x) = \alpha\lambda x^{\lambda-1}\left(1-x^{\lambda}\right)^{\alpha-1}, \qquad 0<x<1; \quad (2)$$

$$F(x) = 1-\left(1-x^{\lambda}\right)^{\alpha}, \quad (3)$$

This distribution is unimodal if $\alpha, \lambda > 1$, uniantimodal if $\alpha, \lambda < 1$, increasing if $\alpha < 1$ and $\lambda > 1$, decreasing if $\lambda < 1$ and $\alpha > 1$ or constant if $\alpha = \lambda = 1$. Figure 1 presents some of these cases for certain values of the shape parameters. In addition, $Kum(\alpha, \lambda)$ provides a large number of well-established distributions, including the Lomax distribution when $\alpha = 1$, the beta type II (inverted beta) distribution when $\lambda = 1$, the log-logistic (Fisk) distribution when $\alpha = \lambda = 1$, the inverted Weibull when $\alpha \to +\infty$, and the generalized exponential when $\lambda \to +\infty$.

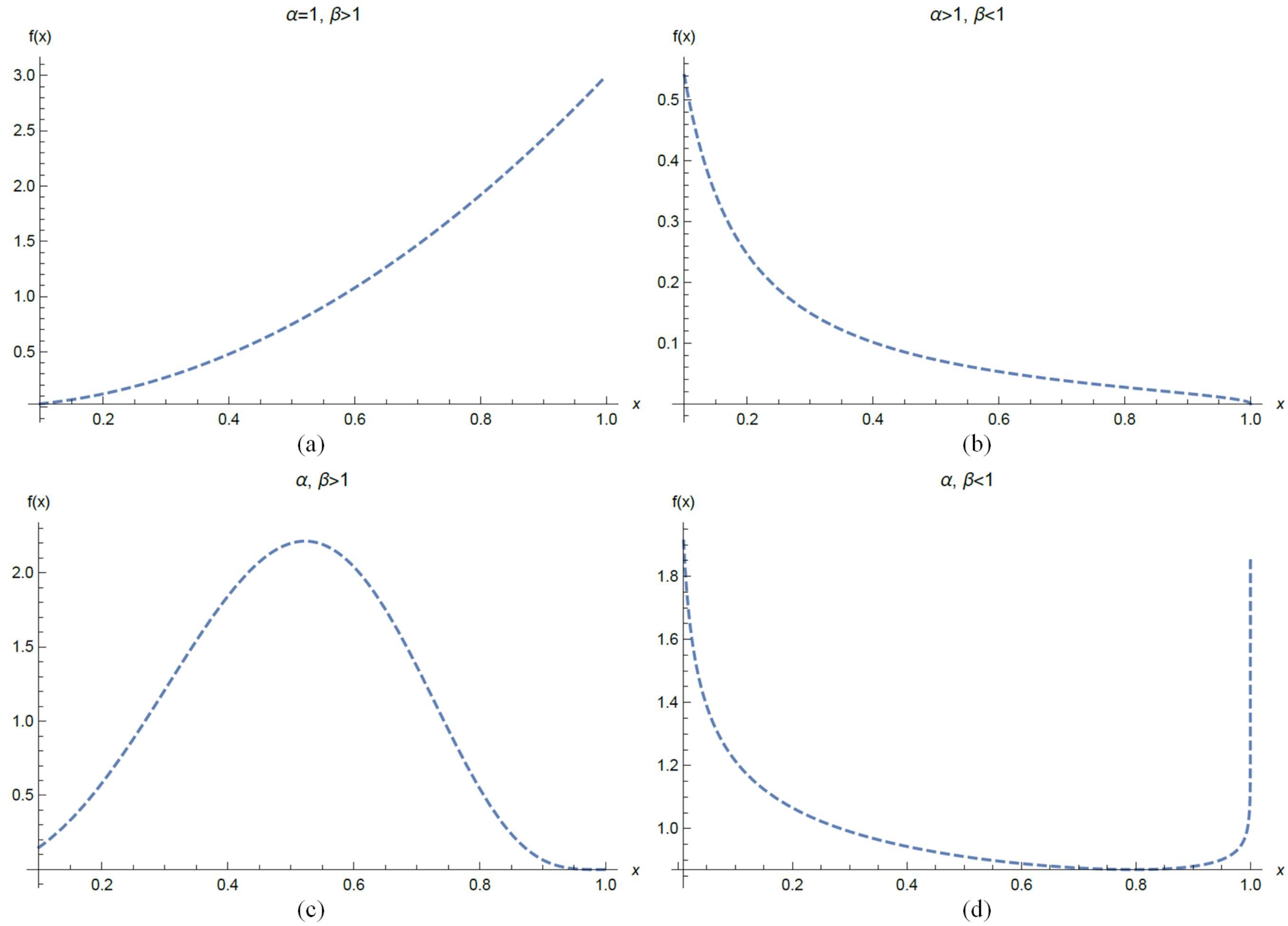

Figure 1. *Shape of Kumaraswamy distribution for various combinations of α and λ.*

The $Kum(\alpha, \lambda)$ is more appropriate for describing hydrological data such as daily rainfall and daily stream flow than the beta distribution (Kumaraswamy [21]; Nadarajah [23]; Jones [18]). Among its other advantages are its tractability under linear transformation as well as exponentiation, the simple formula to generate a random variate, the ability to reproduce Gaussian distributions or extreme value distributions (see Sunder and Subbiah [36]), as well as a simple formula for moments of order statistics, and the ability to fit skewed data not adequately suited to existing distributions (Jones [18]).

The Kumaraswamy distribution has received considerable attention in the literature and has been discussed by many authors, among others, Fletcher and Ponnambalam [12] used $Kum(\alpha, \lambda)$ to model reservoir storage volume. Sundar and Subbiah [36] used $Kum(\alpha, \lambda)$ to fit ocean wave data; Seifi et al. [32] used the $Kum(\alpha, \lambda)$ to model the data taken from a simple voltage divider with two resistors; Ponnambalam et al. [29] used the Kumaraswamy distribution to approximate tolerance ranges for non-symmetric yield distributions.

On the basis of Adaptive-IIPH data, MPS estimators are used to estimating the shape parameters of the Kumaraswamy distribution, and this approach is compared with the conventional MLE. Currently, we are unaware of any literature article that discusses the MPS method to estimate the shape parameters of the Kumaraswamy distribution using the Adaptive-IIPH scheme. Therefore, the main contribution of this study is to propose a robust method for drawing inference about the shape parameters of the Kumaraswamy distribution based on Adaptive-IIPH censored data.

The contents of this article are organized as follows: In Section 2, we introduce the notation and describe the Adaptive-IIPH censoring scheme. A discussion of estimation procedures is presented in Section 3. Here, point estimators have been developed based on MLE and MPS methods with a Kumaraswamy distribution as the underlying lifetime distribution. Section 4 includes the development of asymptotic and Bootstrap confidence intervals based on MLE and MPS under a classical setup. A simulation study is reported in Section 5 which elucidates the performance of the proposed estimator based on the proposed censoring scheme. In Section 6, a real-life data set is analyzed in order to illustrate the proposed methods of estimation. A summary of the results and conclusions of this study is presented in Section 7.

2. Adaptive Type-II Progressive Hybrid Censoring Scheme

Assume there are n units in a life-testing experiment and the effective sample size $m(< n)$ is determined in advance, as well as the censoring scheme $(R_1, R_2..., R_m)$, however, the values of some of the R_i may change as the experiment progresses. Suppose the experimenter provides an ideal total test time T, however, we are allowed to extend the experiment beyond T. If the mth failure occurs before time T (i.e. $X_m < T$), the experiment is carried out in the same way as Progressive Type-II censoring and stops at time X_m with the pre-fixed censoring scheme $(R_1, R_2, ..., R_m)$. Otherwise, if the experimental time has passed T, but the number of observed failures has not yet reached m, we would leave as many surviving units as possible, hoping to see more failures in a short period time, allowing us to complete the experiment in the most effective way possible (see David and Nagaraja [9]), i.e. if $X_j < T < X_{j+1}, j = 0, 1, \ldots, m-1$, we do not withdraw any items from the experiment by setting $R_{j+1} = R_{j+2} = \cdots = R_{m-1} = 0$ and $R_m = n - m - \sum_{i=1}^{j} R_i$. This setting can be seen as a design that guarantees m observed failure times while keeping the total test time not too far away from the ideal test time T (see Figure 1). Note that if $T = 0$, then we have a conventional Type-II censoring scheme, while, if $T \to \infty$, then the Adaptive-IIPH reduces to a Progressive Type-II censoring scheme. If the failure times of the n subjects originally on the test are from a continuous distribution with *cdf* $F(x)$ and pdf $f(x)$, then the likelihood function as given by Ng et al. [24] is:

$$f(x_1, \ldots, x_m) \quad = \quad q_j \prod_{i=1}^{m} f(x_i) \prod_{i=1}^{j} [1 - F(x_i)]^{Ri} \, [1 - F(x_m)]^{R^*}, \tag{4}$$

Where, $R^* = (n - m - \sum_{i=1}^{j} R_i)$, and $q_j = \prod_{i=1}^{m} \left(n - m - \sum_{k=1}^{\min\{i-1,j\}} R_k \right)$

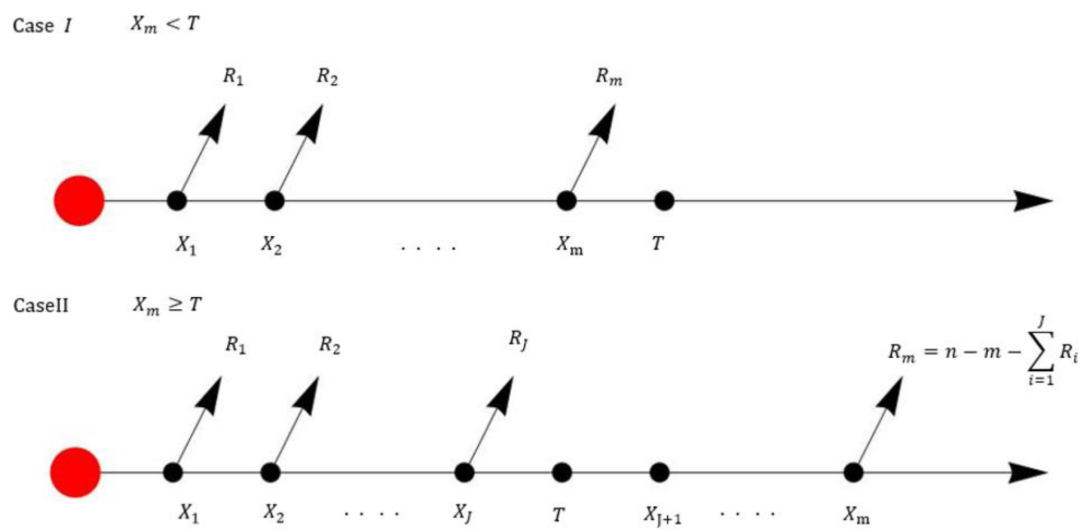

Figure 2. *Schematic representation of Adaptive Type-II Progressive censoring. Case 1: Experiment terminates before time T, Case 2: Experiment terminates after time T.*

3. Parameter Estimation of the Kumaraswamy Distribution

3.1. Maximum Likelihood Method

Assume n independent subjects are tested, and the lifetime distribution of each unit is given by (2). Under the Adaptive-IIPH censoring scheme $\mathbf{R} = (R_1, ..., R_m)$, the ordered m failures are observed. Thus, according to (2), (3) and (4), the log-likelihood function of α and λ based on an Adaptive-IIPH censored data is as follows:

$$\ln L(\alpha,\lambda) \quad \propto \quad m\ln(\alpha\lambda) + (\lambda-1)\sum_{i=1}^{m}\ln(x_i) + (\alpha-1)\sum_{i=1}^{m}\ln(1-x_i^{\lambda}) + \sum_{i=1}^{j}\alpha R_i\ln(1-x_i^{\lambda}) + \alpha R^*\ln(1-x_m^{\lambda}). \quad (5)$$

Then, the $MLEs$ of α and λ, denoted by $\hat{\alpha}_{MLE}$ and $\hat{\lambda}_{MLE}$ respectively are the solutions of the following log-likelihood equations

$$\begin{aligned}\frac{\partial \ln L(\alpha,\lambda)}{\partial\alpha} &= \frac{m}{\alpha} + R^*\ln(1-x_m^{\lambda}) + \sum_{i=1}^{m}\ln(1-x_i^{\lambda}) + \sum_{i=1}^{j}R_i\ln(1-x_i^{\lambda}), \qquad (6)\\ \frac{\partial \ln L(\alpha,\lambda)}{\partial\lambda} &= \frac{m}{\lambda} - \frac{\alpha R^* x_m^{\lambda}\ln x_m}{1-x_m^{\lambda}} + \sum_{i=1}^{m}\ln x_i - (\alpha-1)\sum_{i=1}^{m}\frac{x_i^{\lambda}\ln x_i}{1-x_i^{\lambda}} - \sum_{i=1}^{j}\frac{\alpha R_i x_i^{\lambda}\ln x_i}{1-x_i^{\lambda}}\end{aligned}$$

It is worth noting that there is no explicit closed form solution to Eqs. 6 & 7. As a result, numerical methods based on the SAS/IML language must be used to obtain the $MLEs$ of α and λ.

3.2. Maximum Product of Spacings

Using Pyke (1965) and Eq. (3) the spacing is as follows:

$$\begin{aligned} D_1 &= 1-\left(1-x_1^{\lambda}\right)^{\alpha}, \\ D_{m+1} &= 1-\left(1-x_m^{\lambda}\right)^{\alpha}, \\ D_i &= \left(1-x_{i-1}^{\lambda}\right)^{\alpha} - \left(1-x_i^{\lambda}\right)^{\alpha} \qquad i=1,2,...,m. \qquad (7)\end{aligned}$$

Thus the maximum product of spacings (M) under the Adaptive-IIPH scheme can be written as:

$$M = \prod_{i=1}^{m+1} D_i \prod_{i=1}^{j}\left[1-F(x_i)\right]^{R_i}\left[1-F(x_m)\right]^{R^*},$$

Using Eqs. (3) and (7) we get:

$$M = \left[1-\left(1-x_1^{\lambda}\right)^{\alpha}\right]\left(1-x_m^{\lambda}\right)^{\alpha(R^*+1)}\prod_{i=2}^{m}\left[\left(1-x_{i-1}^{\lambda}\right)^{\alpha}-\left(1-x_i^{\lambda}\right)^{\alpha}\right]\times\prod_{i=1}^{j}\left(1-x_i^{\lambda}\right)^{\alpha R_i}.$$

With log-likelihood function:

$$\ln M = \ln[1-(1-x_1^{\lambda})^{\alpha}] + \alpha(R^*+1)\ln(1-x_m^{\lambda}) + \sum_{i=2}^{m}\ln\left[(1-x_{i-1}^{\lambda})^{\alpha}-(1-x_i^{\lambda})^{\alpha}\right] + \sum_{i=1}^{j}\alpha R_i\ln(1-x_i^{\lambda}). \quad (8)$$

The partial derivatives of Eq. (8) with respect to the unknown parameters are given as follows:

$$\begin{aligned}\frac{\partial \ln M}{\partial\alpha} &= \frac{-\left(1-x_1^{\lambda}\right)^{\alpha}\ln\left(1-x_1^{\lambda}\right)}{1-\left(1-x_1^{\lambda}\right)^{\alpha}} + (R^*+1)\ln\left(1-x_m^{\lambda}\right) + \sum_{i=1}^{j}R_i\ln\left(1-x_i^{\lambda}\right)\\ &\quad + \sum_{i=2}^{m}\frac{\left(1-x_{i-1}^{\lambda}\right)^{\alpha}\ln(1-x_{i-1}^{\lambda}) - \left(1-x_i^{\lambda}\right)^{\alpha}\ln(1-x_i^{\lambda})}{\left(1-x_{i-1}^{\lambda}\right)^{\alpha}-\left(1-x_i^{\lambda}\right)^{\alpha}}, \qquad (9)\end{aligned}$$

$$\frac{\partial \ln M}{\partial \lambda} = \frac{\alpha x_1^{\lambda}\left(1-x_1^{\lambda}\right)^{\alpha-1}\ln x_1}{1-\left(1-x_1^{\lambda}\right)^{\alpha}} - \frac{\alpha\left(R^*+1\right)x_m^{\lambda}\ln x_m}{1-x_m^{\lambda}} - \sum_{i=1}^{j}\frac{\alpha x_i^{\lambda}R_i\ln x_i}{1-x_i^{\lambda}}$$
$$-\alpha\sum_{i=2}^{m}\frac{x_{i-1}^{\lambda}(1-x_{i-1}^{\lambda})^{\alpha-1}\ln x_{i-1}-x_i^{\lambda}\left(1-x_i^{\lambda}\right)^{\alpha-1}\ln x_i}{\left(1-x_{i-1}^{\lambda}\right)^{\alpha}-\left(1-x_i^{\lambda}\right)^{\alpha}}. \tag{10}$$

The MPS of the model parameters are the solutions of those non-linear equations after setting them equal to zero. Because there are no explicit solutions to these equations, iterative approaches using SAS/IML language are used.

4. Interval Estimation

Let $x_1 \leq x_2 \leq x_j \leq \cdots \leq x_{m-1} \leq x_m$ denote an Adaptive-IIPH censored sample from the Kumaraswamy distribution with parameters α and λ. The two types of interval estimation methods for the parameters of the Kumaraswamy distribution, that we shall explore in this section are the asymptotic and the bootstrap confidence intervals which was proposed by Efron [10].

4.1. Asymptotic Confidence Intervals (A.CI)

In this subsection, the asymptotic confidence intervals for the parameters of the $Kum(\alpha,\lambda)$ using the MLE and the MPS methods based on the Adaptive-IIPH scheme will be investigated. The interval estimation of the parameters requires the variance-covariance matrix, which is the approximate inverse of the Fisher information matrix

$$I^{-1}\left(\hat{\alpha},\hat{\lambda}\right) = \begin{bmatrix} -\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial\alpha^2} & -\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial\alpha\partial\lambda} \\ -\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial\lambda\partial\alpha} & -\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial\lambda^2} \end{bmatrix}^{-1}_{(\alpha,\lambda)=\left(\hat{\alpha},\hat{\lambda}\right)} = \begin{bmatrix} var\left(\hat{\alpha}\right) & cov\left(\hat{\alpha},\hat{\lambda}\right) \\ cov\left(\hat{\lambda},\hat{\alpha}\right) & var\left(\hat{\lambda}\right) \end{bmatrix}.$$

Based on regularity conditions, $\left(\hat{\alpha}_{MLE},\hat{\lambda}_{MLE}\right) \approx Normal\left(\left(\alpha,\lambda\right), I^{-1}\left(\hat{\alpha}_{ML},\hat{\lambda}_{ML}\right)\right)$, where, $I\left(\hat{\alpha},\hat{\lambda}\right)$ is the observed information matrix. From Eqs. 6 & 7, we can easily get

$$\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial\alpha^2} = -\frac{m}{\alpha^2}, \tag{11}$$

$$\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial\lambda^2} = -\frac{m}{\lambda^2} - \frac{\alpha R^*\, x_m^{\lambda}\,\left(\ln x_m\right)^2}{\left(1-x_m^{\lambda}\right)^2} + (\alpha-1)\sum_{i=1}^{m}\frac{x_i^{\lambda}\,\left(\ln x_i\right)^2}{\left(1-x_i^{\lambda}\right)^2} - \sum_{i=1}^{j}\frac{\alpha R_i\, x_i^{\lambda}\,\left(\ln x_i\right)^2}{\left(1-x_i^{\lambda}\right)^2}, \tag{12}$$

$$\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial\alpha\partial\lambda} = -\frac{R^*\, x_m^{\lambda}\,\ln x_m}{1-x_m^{\lambda}} - \sum_{i=1}^{m}\frac{x_i^{\lambda}\,\ln x_i}{1-x_i^{\lambda}} - \sum_{i=1}^{j}\frac{R_i x_i^{\lambda}\,\ln x_i}{1-x_i^{\lambda}}. \tag{13}$$

Thus, the $100\left(1-\gamma\right)\%$ asymptotic two-sided confidence intervals for $\alpha\&\lambda$ are, respectively, given by: $\left(\hat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{var\left(\hat{\alpha}\right)}\right)$ and $\left(\hat{\lambda} \pm Z_{\frac{\gamma}{2}}\sqrt{var\left(\hat{\lambda}\right)}\right)$, where $Z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}th$ percentile of the standard normal distribution.

Using the same algorithm, one can obtain the Fisher information matrix for the estimators of the MPS method and subsequently the confidence intervals for α and λ.

4.2. Parametric Bootstrap Confidence Interval (B.CI)

A parametric bootstrap interval, as opposed to a point estimate, provides substantially more information about the population value of the quantity of interest. Here we construct the parametric bootstrap for $\theta = (\alpha,\lambda)$ by using the percentile bootstrap method which is demonstrated as follows.

1. Based on the original sample $\mathbf{x} = \{x_1, x_2, \ldots, x_j, \ldots, x_{m-1}, x_m\}$ calculate the estimator of $\theta = (\alpha,\lambda)$ and name it $\hat{\theta}_0^* = \left(\hat{\alpha}_0^*, \hat{\lambda}_0^*\right)$.
2. Use $\hat{\theta}_0^*$ to generate a new bootstrap sample $\left\{x_{11}^*, x_{21}^*, \ldots, x_{J1}^*, \ldots, x_{m-1}^*, x_{m1}^*\right\}$, then use this new sample to calculate a new estimator $\hat{\theta}_1^* = \left(\hat{\alpha}_1^*, \hat{\lambda}_1^*\right)$.
3. Step 2 should be repeated B times, where B is a large number, e.g. $B = 5000$. At this point a set of estimates have been obtained: $\hat{\theta}^* = \left(\hat{\theta}^{*1}, \hat{\theta}^{*2}, \ldots, \hat{\theta}^{*B}\right)$
4. Sort $\hat{\alpha}_i^*$ and $\hat{\lambda}_i^*$ in an ascending order, respectively. Then, we have:

$$\begin{cases} \hat{\alpha}_i^* = \left(\hat{\alpha}^{*(1)}, \hat{\alpha}^{*(2)}, \ldots, \hat{\alpha}^{*(B)}\right) \\ \hat{\lambda}_i^* = \left(\hat{\lambda}^{*(1)}, \hat{\lambda}^{*(2)}, \ldots, \hat{\lambda}^{*(B)}\right) \end{cases}$$

5. The approximation $100\,(1-\gamma)\,\%$ confidence intervals for α and λ are given, respectively, by $\left[\hat{\alpha}^{*\left(\frac{B\gamma}{2}\right)}, \hat{\alpha}^{*\left(1-\frac{B\gamma}{2}\right)}\right]$ & $\left[\hat{\lambda}^{*\left(\frac{B\gamma}{2}\right)}, \hat{\lambda}^{*\left(1-\frac{B\gamma}{2}\right)}\right]$

5. Simulation Study

In this section, we undertake a simulation study to test the performance of the various estimation methods that we have discussed previously. We generate progressively censored samples (Balakrishnan and Cramer (2014)) from the Kumaraswamy distribution as follows:

1. Generate m independent $U(0,1)$ random variables $W_1, W_2, ..., W_m$.
2. For given values of Progressive censoring scheme $R_1, R_2, ..., R_m$, we set $E_i = (i + \sum\limits_{j=m-i+1}^{m} R_j), i = 1, ..., m$. $V_i = W_i^{1/E_i}$, for $i = 1, ..., m$.
3. Consider $u_i = 1 - V_m \times V_{m-1} \times ... \times V_{m-i+1}$, $i = 1, ..., m$, then $u_1, ..., u_m$ is a Progressive Type-II censored sample of size m from $U(0,1)$.
4. For given values of α and λ we set $x_i = F^{-1}(u_i) = \left[1-(1-u_i)^{\frac{-1}{\alpha}}\right]^{\frac{1}{\lambda}}, i = 1, ..., m$. Finally, $x_1, x_2, ..., x_m$ is the required Progressive Type-II censored sample of size m from the $Kum(\alpha, \lambda)$ distribution.
5. Determine the value of j, where $x_j < T < x_{j+1}$ and discard the sample $x_{j+2}, \ldots, x_m$.
6. Generate the first $m-j-1$ order statistics from a truncated distribution $\frac{f(x)}{1-F(x_{j+1})}$ with sample size $n - \sum\limits_{i=1}^{j} R_{i-j-1}$ as $x_{j+2}, \ldots, x_m$.
7. Obtain the MLE and the MPS estimates of the model parameters using iterative process.

We generate 5000 Adaptive-IIPH censored samples from the Kumaraswamy distribution with $(\alpha, \lambda) = (0.7, 0.7); (1.5, 3)$, two different T values: $T_1 = X_{\frac{4*m}{5}}$; and $T_2 = (X_m + 2)$ and different combinations of sample sizes and effective sample sizes (n, m) : $(n, m) = (20, 14); (40, 30); (60, 50); (200, 140)$ are conducted with three different censoring schemes $(R_1, ..., R_m)$. For simplicity of notations, $R = (0^{*4})$ indicates $R = (0, 0, 0, 0)$. The three censoring schemes are shown below:

1. Censoring scheme (Cs) I : $R_1 = n - m, R_i = 0$, for $i \neq 1$.
2. Censoring scheme (Cs) II : $R_m = n - m, R_i = 0, R_i = 0$ for $i \neq m$.
3. Censoring scheme (Cs) III : $R_1 = R_m = (n - m)/2, R_i = 0$ for $i \neq 1$ & $i \neq m$.

The performance of the MLE and the MPS estimates for α and λ is compared in terms of their absolute bias (*Bias*) and their mean square error (*MSE*). Suppose $\hat{\theta}_i$ is the estimate of θ for the *i-th* simulated data set, then the *Bias* and the *MSE* are computed as follows: $Bias = \frac{1}{5000}\sum\limits_{i=1}^{5000}\left|\hat{\theta}_i - \theta\right|$ & $MSE = \frac{1}{5000}\sum\limits_{i=1}^{5000}(\hat{\theta}_i - \theta)^2$.

In addition, we compute 95% asymptotic confidence intervals and symmetric bootstrap confidence intervals based on 1000 bootstrap samples. We repeat the process 5000 times and obtain the average lengths (L) of the confidence intervals. All values are reported in Tables 1 - 4.

1. As shown in Tables 1 & 2, as the effective sample size m increases, all estimators exhibit the property of consistency, which means their MSE values approach zero.
2. In addition, a smaller $Bias$ is observed for the MLEs as compared to the MPS when $X_m < T$. However, "the opposite" holds true when $X_m > T$ and the MPS-$Bias$ is larger than the MLEs.
3. The study also depicts that the estimates based on MPS outperform the estimates based on the MLE in terms of MSE values for all values of T, m, n and different censoring schemes, demonstrating that the MPS technique is useful in estimating the shape parameters of the Kumaraswamy distribution.
4. For fixed n, m, and T, estimates based on Scheme I perform better than the ones based on Schemes II, and III, in terms of $Bias$ and MSE values.
5. It should also be noted that the $Bias$ of all MPS-based estimates and MlE-based estimates, decrease as the effective sample size m increases, as expected.
6. Overall, the simulation results suggest that with a large effective sample size $m(m \geq 50)$, the differences in MSE values between MLE and MPS methods of estimation become minimal.

While Tables 1- 4 show *Bias* and *MSE* values for each estimating method. Thus, it is crucial to understand how each estimation method handles interval estimation. As a consequence, at 0.95 confidence levels, we generate asymptotic as well as parametric Bootstrap confidence intervals. We calculated the average length of these intervals and present our findings in Tables 3 and 4. Based on these tables, the following conclusions can be drawn.

1. The Bootstrap confidence intervals have shorter average length than the asymptotic confidence intervals. Further more, the average length is narrower as effective sample size m increases.
2. The confidence intervals based on MPS provides smaller length as compared to the the MLE-based estimators. Moreover, the average length of the confidence intervals when $T < X_m$ is slightly shorter than when $T > X_m$.
3. The differences in terms of average interval length between different censoring schemes is minimal. This indicates that the censoring scheme has no effect on confidence intervals.

Table 1. *$Bias$ and MSE of the estimators of α and λ under different censoring schemes when $\alpha = 0.7$, $\lambda = 0.7$.*

				$T_1 = X_{[\frac{4m}{5}]}$				$T_2 = X_{[m]+2}$			
n	m	SC	Estimation	α		λ		α		λ	
			Method	$Bias$	MSE	$Bias$	MSE	$Bias$	MSE	$Bias$	MSE
20	14	I	MLE	0.0516	0.00622	0.0590	0.0062	0.1603	0.01133	0.1124	0.0080
			MPS	0.1157	0.00413	0.1275	0.0049	0.0396	0.00481	0.0845	0.0051
		II	MLE	0.0743	0.00868	0.0697	0.0065	0.2289	0.02270	0.1421	0.0100
			MPS	0.1167	0.00489	0.1261	0.0051	0.0263	0.00925	0.0446	0.0060
		III	MLE	0.0646	0.00768	0.0621	0.0061	0.1901	0.01543	0.1219	0.0082
			MPS	0.1212	0.00465	0.1293	0.0049	0.0382	0.00581	0.0818	0.0051
40	30	I	MLE	0.0203	0.00108	0.0355	0.0012	0.0716	0.00139	0.0653	0.0015
			MPS	0.0755	0.00093	0.0796	0.0011	0.0354	0.00087	0.0541	0.0011
		II	MLE	0.0312	0.00135	0.0386	0.0012	0.0922	0.00200	0.0740	0.0016
			MPS	0.0762	0.00106	0.0777	0.0010	0.0070	0.00122	0.0011	0.1212
		III	MLE	0.0271	0.00125	0.0346	0.0011	0.0831	0.00171	0.0676	0.0014
			MPS	0.0792	0.00103	0.0798	0.0010	0.0349	0.00101	0.0515	0.0011
60	50	I	MLE	0.0134	0.00031	0.0199	0.0004	0.0449	0.00044	0.0386	0.0005
			MPS	0.0548	0.00030	0.0616	0.0004	0.0270	0.00033	0.0446	0.0004
		II	MLE	0.0159	0.00035	0.0204	0.0004	0.0514	0.00053	0.0411	0.0005
			MPS	0.0556	0.00032	0.0605	0.0004	0.0055	0.00039	0.0290	0.0004
		III	MLE	0.0145	0.00034	0.0186	0.0004	0.0484	0.00049	0.0384	0.0005
			MPS	0.0572	0.00032	0.0617	0.0004	0.0277	0.00035	0.0436	0.0004
200	140	I	MLE	0.0020	0.00004	0.0037	0.0001	0.0126	0.00004	0.0101	0.0001
			MPS	0.0291	0.00004	0.0335	0.0001	0.0190	0.00004	0.0274	0.0001
		II	MLE	0.0032	0.00005	0.0038	0.0000	0.0161	0.00006	0.0112	0.0001
			MPS	0.0313	0.00005	0.0318	0.0001	0.0066	0.00005	0.0166	0.0001
		III	MLE	0.0023	0.00004	0.0028	0.0000	0.0097	0.00050	0.0097	0.0000
			MPS	0.0315	0.00004	0.0318	0.0000	0.0025	0.00050	0.0252	0.0000

Table 2. *$Bias$ and MSE of the estimators of α and λ under different censoring schemes when $\alpha = 1.5$, $\lambda = 3$.*

				$T_1 = X_{[\frac{4m}{5}]}$				$T_2 = X_{[m]+2}$			
n	m	SC	Estimation	α		λ		α		λ	
			Method	$Bias$	MSE	$Bias$	MSE	$Bias$	MSE	$Bias$	MSE
20	14	I	MLE	0.1612	0.04474	0.1593	0.0687	0.4509	0.09408	0.3613	0.0875
			MPS	0.2816	0.02476	0.5005	0.0644	0.0981	0.03238	0.3372	0.0623
		II	MLE	0.2448	0.07103	0.2080	0.0749	0.6097	0.19057	0.4275	0.0988
			MPS	0.2799	0.03124	0.4958	0.0670	0.0708	0.05062	0.3225	0.0660
		III	MLE	0.2052	0.05858	0.1747	0.0688	0.5495	0.14619	0.3973	0.0903
			MPS	0.2908	0.02862	0.5043	0.0646	0.0872	0.04275	0.3265	0.0625
40	30	I	MLE	0.0686	0.00697	0.0968	0.0136	0.2007	0.01001	0.2157	0.0173
			MPS	0.1878	0.00546	0.3086	0.0138	0.0867	0.00553	0.2063	0.0138
		II	MLE	0.0984	0.00933	0.1139	0.0139	0.2542	0.01459	0.2392	0.0183
			MPS	0.1903	0.00651	0.3044	0.0138	0.0756	0.00722	0.1968	0.0142
		III	MLE	0.0821	0.00819	0.0956	0.0129	0.2343	0.01270	0.2245	0.0169
			MPS	0.1963	0.00611	0.3093	0.0133	0.0831	0.00655	0.1981	0.0135
60	50	I	MLE	0.0364	0.00195	0.0459	0.0051	0.1192	0.00292	0.1201	0.0060
			MPS	0.1413	0.00179	0.2398	0.0055	0.0712	0.00201	0.1719	0.0055
		II	MLE	0.0454	0.00230	0.0515	0.0051	0.1360	0.00355	0.1275	0.0061
			MPS	0.1445	0.00201	0.2374	0.0054	0.0689	0.00233	0.1678	0.0055
		III	MLE	0.0395	0.00214	0.0429	0.0048	0.1288	0.00329	0.1205	0.0058
			MPS	0.1471	0.00194	0.2402	0.0053	0.0724	0.00220	0.1691	0.0053
200	140	I	MLE	0.0057	0.00022	0.0059	0.0006	0.0332	0.00027	0.0318	0.0007
			MPS	0.0755	0.00023	0.1255	0.0007	0.0499	0.00025	0.1004	0.0007
		II	MLE	0.0102	0.00029	0.0082	0.0006	0.0415	0.00036	0.0343	0.0006
			MPS	0.0824	0.00029	0.1224	0.0007	0.0534	0.00031	0.0969	0.0007
		III	MLE	0.0067	0.00026	0.0036	0.0006	0.0376	0.00032	0.0307	0.0006
			MPS	0.0815	0.00027	0.1209	0.0006	0.0529	0.00028	0.0943	0.0006

Table 3. *The average length of the asymptotic and bootstrap intervals under different censoring schemes using* $\alpha = 0.7, \lambda = 0.7$.

				$T_1 = X_{[\frac{4m}{5}]}$				$T_2 = X_{[m]+2}$			
n	n	**SC**	**Estimation**	α		λ		$\boldsymbol{\alpha}$		$\boldsymbol{\lambda}$	
			Method	$\boldsymbol{LN}$	$\boldsymbol{LB}$	$\boldsymbol{LN}$	$\boldsymbol{LB}$	$\boldsymbol{LN}$	$\boldsymbol{LB}$	$\boldsymbol{LN}$	$\boldsymbol{LB}$
20	14	I	MLE	0.26377	0.0046788	0.27552	0.0050550	0.30983	0.0039980	0.28618	0.0047686
			MPS	0.20059	0.0035203	0.22250	0.0047315	0.23103	0.0045224	0.23131	0.0047671
		II	MLE	0.29919	0.0048243	0.27647	0.0049233	0.39537	0.0041244	0.30123	0.0046167
			MPS	0.21640	0.0035219	0.22582	0.0042277	0.29313	0.0045363	0.25013	0.0043130
		III	MLE	0.28628	0.0047965	0.26962	0.0049139	0.34525	0.0040947	0.28101	0.0045851
			MPS	0.21017	0.0034658	0.22213	0.0043048	0.24668	0.0044958	0.23194	0.0043546
40	30	I	MLE	0.11678	0.0024650	0.13013	0.0027368	0.12633	0.0022748	0.13339	0.0026289
			MPS	0.09956	0.0020873	0.11421	0.0023973	0.10707	0.0024633	0.11704	0.0023901
		II	MLE	0.12805	0.0025393	0.12833	0.0026539	0.14448	0.0023489	0.13440	0.0025324
			MPS	0.10686	0.0021008	0.11398	0.0021336	0.12538	0.0024789	0.12115	0.0021428
		III	MLE	0.12407	0.0025231	0.12553	0.0026426	0.13588	0.0023304	0.12884	0.0025155
			MPS	0.10415	0.0020697	0.11210	0.0021732	0.11319	0.0024486	0.11504	0.0021677
60	50	I	MLE	0.06906	0.0015620	0.07876	0.0017265	0.07273	0.0014535	0.07995	0.0016588
			MPS	0.06185	0.0013674	0.07205	0.0014647	0.06501	0.0015632	0.07315	0.0014609
		II	MLE	0.07284	0.0015981	0.07753	0.0016863	0.07808	0.0014918	0.07952	0.0016122
			MPS	0.06470	0.0013746	0.07139	0.0013514	0.07149	0.0015714	0.07404	0.0013531
		III	MLE	0.07159	0.0015904	0.07648	0.0016792	0.07574	0.0014830	0.07768	0.0016038
			MPS	0.06370	0.0013606	0.07060	0.0013681	0.06724	0.0015582	0.07172	0.0013646
200	140	I	MLE	0.02414	0.0005696	0.02754	0.0006753	0.02458	0.0005471	0.02770	0.0006643
			MPS	0.02296	0.0005296	0.02639	0.0005265	0.02337	0.0005745	0.02655	0.0005251
		II	MLE	0.02657	0.0006102	0.02609	0.0006299	0.02772	0.0005867	0.02673	0.0006176
			MPS	0.02508	0.0005257	0.02517	0.0004428	0.02673	0.0005710	0.02597	0.0004437
		III	MLE	0.02572	0.0005994	0.02543	0.0006217	0.02626	0.0005754	0.02557	0.0006084
			MPS	0.02435	0.0005229	0.02460	0.0004521	0.02485	0.0005696	0.02474	0.0004508

Table 4. *The average length of the asymptotic and bootstrap intervals under different censoring schemes using* $\alpha = 1.5, \lambda = 3$.

				$T_1 = X_{[\frac{4m}{5}]}$				$T_2 = X_{[m]+2}$			
n	m	**SC**	**Estimation**	$\boldsymbol{\alpha}$		$\boldsymbol{\lambda}$		$\boldsymbol{\alpha}$		$\boldsymbol{\lambda}$	
			Method	$\boldsymbol{LN}$	$\boldsymbol{LB}$	$\boldsymbol{LN}$	$\boldsymbol{LB}$	$\boldsymbol{LN}$	$\boldsymbol{LB}$	$\boldsymbol{LN}$	$\boldsymbol{LB}$
20	14	I	MLE	0.67508	0.0463168	0.95338	0.0478844	0.82674	0.1038017	0.99174	0.0542365
			MPS	0.46818	0.0187779	0.79471	0.0321290	0.55372	0.0329019	0.82296	0.0376888
		II	MLE	0.80490	0.0681241	0.96874	0.0500195	1.03658	0.2202268	1.01249	0.0579381
			MPS	0.51939	0.0215482	0.81269	0.0327877	0.63352	0.0398694	0.84421	0.0390457
		III	MLE	0.74768	0.0579993	0.93554	0.0477004	0.94799	0.1770489	0.97559	0.0550062
			MPS	0.49750	0.0203020	0.79494	0.0318790	0.60220	0.0375221	0.82530	0.0380913
40	30	I	MLE	0.29074	0.0151283	0.45196	0.0223604	0.32217	0.0186226	0.46416	0.0235624
			MPS	0.23554	0.0096344	0.40537	0.0180272	0.25813	0.0113842	0.41543	0.0188498
		II	MLE	0.32920	0.0185354	0.45297	0.0227821	0.37238	0.0232336	0.46608	0.0240758
			MPS	0.25838	0.0109468	0.40846	0.0182726	0.28793	0.0130451	0.41909	0.0191281
		III	MLE	0.31172	0.0167960	0.43800	0.0218123	0.35085	0.0211704	0.45016	0.0231063
			MPS	0.24836	0.0102919	0.39834	0.0177136	0.27579	0.0123148	0.40849	0.0185910
60	50	I	MLE	0.17009	0.0082009	0.27359	0.0130897	0.18179	0.0090513	0.27805	0.0135821
			MPS	0.14718	0.0061061	0.25418	0.0113569	0.15669	0.0066989	0.25815	0.0117709
		II	MLE	0.18302	0.0091069	0.27291	0.0131306	0.19715	0.0100858	0.27754	0.0136320
			MPS	0.15633	0.0065913	0.25416	0.0113852	0.16759	0.0072627	0.25824	0.0118086
		III	MLE	0.17724	0.0086629	0.26685	0.0127704	0.19049	0.0096233	0.27127	0.0132760
			MPS	0.15232	0.0063471	0.24958	0.0111351	0.16298	0.0070077	0.25352	0.0115662
200	140	I	MLE	0.05907	0.0027093	0.09669	0.0043394	0.06043	0.0028018	0.09728	0.0043986
			MPS	0.05530	0.0023604	0.09336	0.0040711	0.05655	0.0024393	0.09393	0.0041252
		II	MLE	0.06687	0.0031060	0.09389	0.0042311	0.06869	0.0032247	0.09448	0.0042900
			MPS	0.06188	0.0026451	0.09087	0.0039693	0.06354	0.0027434	0.09144	0.0040238
		III	MLE	0.06325	0.0029204	0.09013	0.0040536	0.06489	0.0030323	0.09067	0.0041124
			MPS	0.05895	0.0025161	0.08760	0.0038226	0.06045	0.0026106	0.08812	0.0038775

6. Real Life Data

In this section, we consider a real life data to demonstrate the proposed method and verify how our estimates work in practice. The data for this application were utilized by Brito [5]. The data cover the milk production of SINDI cows during the period 1987 to 1997. The data set does not belong to the interval (0,1), thus we transform the data using the the following equation $x_i = \frac{y_i - \min(y_i)}{\max(y_i) - \min(y_i)}$, for $i = 1, ..., 107$. The transformed data is given in Table 5.

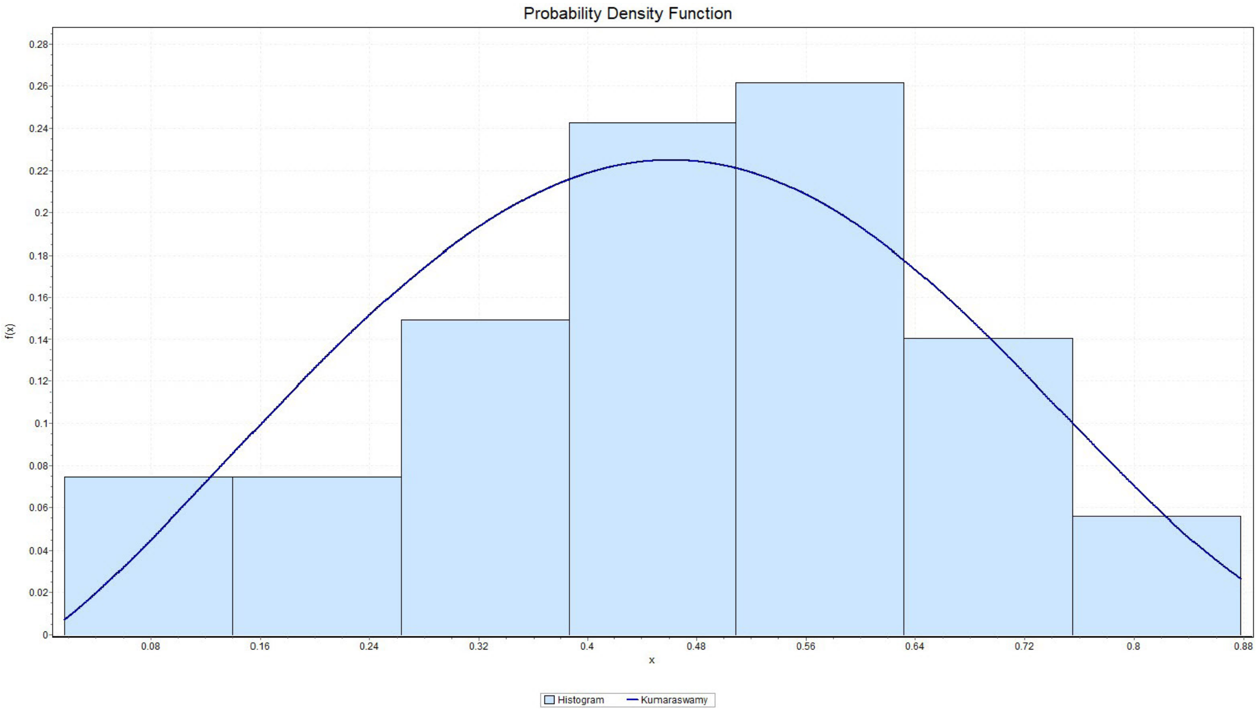

Figure 3. *The histogram of the data set and its fitted density function to the milk production data.*

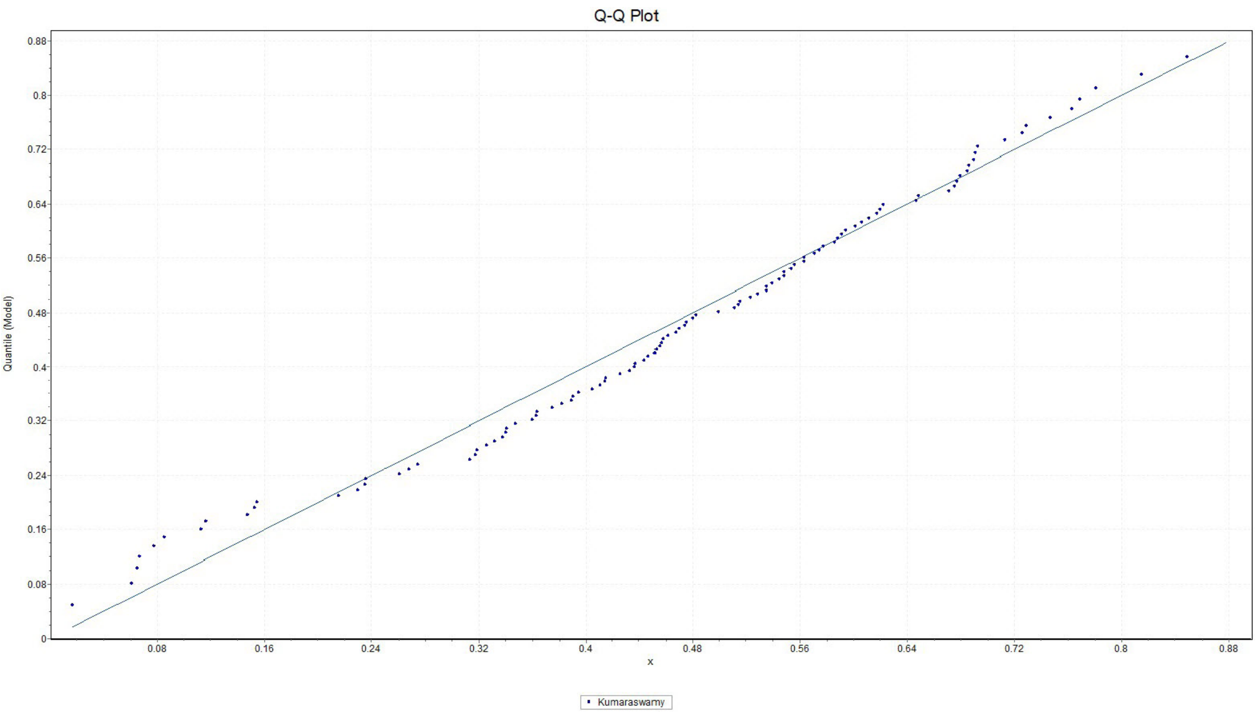

Figure 4. *Plot of the empirical quantile of Kumaraswamy distribution fitted to the milk production data.*

Table 5. *Proportion of total milk production.*

0.4365	0.4260	0.5140	0.6907	0.7471	0.2605	0.6196	0.3945	0.4553	0.3598	0.7629	0.3635
0.8781	0.4990	0.6058	0.6891	0.5770	0.5394	0.1479	0.4470	0.5285	0.5941	0.6174	0.4694
0.2356	0.6012	0.1525	0.5483	0.6927	0.7261	0.3323	0.5232	0.6465	0.6860	0.0609	0.3821
0.0671	0.2361	0.4800	0.5707	0.7131	0.5853	0.6768	0.0650	0.8492	0.6488	0.2747	0.0854
0.5350	0.4151	0.6789	0.4576	0.3259	0.2303	0.7687	0.8147	0.3627	0.5349	0.3751	0.4332
0.4371	0.3383	0.6114	0.3480	0.4564	0.7804	0.3406	0.3906	0.4438	0.1546	0.4517	0.3413
0.4823	0.5912	0.5744	0.5481	0.1131	0.7290	0.0168	0.4612	0.3188	0.2681	0.4049	0.6844
0.5529	0.4530	0.3891	0.4752	0.3134	0.3175	0.1167	0.2160	0.6707	0.5553	0.5878	0.4675
0.6750	0.5113	0.5447	0.4143	0.5627	0.5150	0.0776	0.6220	0.5629	0.4741	0.4111	

The legitimacy of the Kumaraswamy model is checked based on $\alpha = 3.436$ and $\lambda = 2.1949$ using Kolmogrov-Smirnov (K-S) test, as well as Anderson-Darling (A-D) and chi-square tests. It is observed that K-$S = 0.07625$ with $p_{value} = 0.5372$, A-$D = 1.003$ and chi-square distance $= 5.0832$ with a corresponding $p_{value} = 0.53318$. This indicates that the Kumaraswamy model provides a good fit to the above data. In addition, Figure 3 gives the histogram of the data-set and the plots of the fitted density. The QQ plot in Figure 4 suggests that the Kumaraswamy distribution is very suitable for the milk production data.

The following artificial Adaptive-IIHP censored data are generated from this data using the same censoring schemes as those described in Section 6, and they are listed below:

1. *Scheme I* : $n = 107$, $m = 93$, $R = \{14, 0^{*92}\}$.
2. *Scheme II* : $n = 107$, $m = 93$, $R = \{0^{*92}, 14\}$.
3. *Scheme III* : $n = 107$, $m = 93$, $R = \{7, 0^{*91}, 7\}$.

Table 6 displays the generated data. The calculated MLE and MPS estimates, using Scheme I - Scheme III in combination with T_1 and T_2, are listed in Table 7. Results for T_3 are similar.

We performed all calculations using SAS/IML. Based on time T_1, schemes I, II, and III each needed 14.85 seconds, 13.50 seconds, and 13.64 seconds, respectively, to converge. While they took 16.38s, 16.69s, and 13.35s, respectively, to converge when using T_2.

Table 6. *The meteorological study data.*

Scheme	Censored Data
I	0.0168, 0.0671, 0.0776, 0.0854, 0.1131, 0.1167, 0.1479, 0.1525, 0.1546, 0.2160, 0.2303, 0.2356, 0.2361, 0.2605, 0.2681, 0.2747, 0.3134, 0.3175, 0.3188, 0.3259, 0.3323, 0.3383, 0.3406, 0.3413, 0.348, 0.3598, 0.3627, 0.3635, 0.3751, 0.3821, 0.3891, 0.3906, 0.3945, 0.4049, 0.4111, 0.4143, 0.4151, 0.4260, 0.4332, 0.4365, 0.4371, 0.4438, 0.4470, 0.4517, 0.4530, 0.4553, 0.4564, 0.4576, 0.4612, 0.4675, 0.4694, 0.4741, 0.4752, 0.4800, 0.4823, 0.4990, 0.5113, 0.5140, 0.515 , 0.5232, 0.5285, 0.5349, 0.535, 0.5394, 0.5447, 0.5481, 0.5483, 0.5529, 0.5553, 0.5627, 0.5629, 0.5707, 0.5744,0.5770, 0.5853, 0.5878, 0.5912, 0.5941, 0.6012, 0.6058, 0.6114, 0.6174, 0.6196, 0.6220, 0.6465, 0.6488, 0.6707, 0.6750, 0.7471, 0.7804, 0.8147, 0.8492, 0.8781
II	0.0168, 0.0609, 0.0650, 0.0671, 0.0776, 0.0854, 0.1131, 0.1167, 0.1479, 0.1525, 0.1546, 0.216, 0.2303, 0.2356, 0.2361, 0.2605, 0.2681, 0.2747, 0.3134, 0.3175, 0.3188, 0.3259, 0.3323, 0.3383, 0.3406, 0.3413, 0.3480, 0.3598, 0.3627, 0.3635, 0.3751, 0.3821, 0.3891, 0.3906, 0.3945, 0.4049, 0.4111, 0.4143, 0.4151, 0.4260, 0.4332, 0.4365, 0.4371, 0.4438, 0.447, 0.4517, 0.4530, 0.4553, 0.4564, 0.4576, 0.4612, 0.4675, 0.4694, 0.4741, 0.4752, 0.4800, 0.4823, 0.4990, 0.5113, 0.5140, 0.5150 , 0.5232, 0.5285, 0.5349, 0.5350, 0.5394, 0.5447, 0.5481, 0.5483, 0.5529, 0.5553, 0.5627, 0.5629, 0.5707, 0.5744, 0.5770, 0.5853, 0.5878, 0.5912, 0.5941, 0.6012, 0.6058, 0.6114, 0.6174, 0.6196, 0.6220, 0.6465, 0.6488, 0.6707, 0.6750, 0.6768, 0.6789, 0.6844
III	0.0168, 0.0671, 0.0776, 0.0854, 0.1131, 0.1167, 0.1479, 0.1525, 0.1546, 0.2160, 0.2303, 0.2356, 0.2681, 0.2747, 0.3134, 0.3175, 0.3188, 0.3259, 0.3323, 0.3383, 0.3406, 0.3413, 0.3480, 0.3635, 0.3751, 0.3821, 0.3891, 0.3906, 0.3945, 0.4049, 0.4111, 0.4143, 0.4151, 0.4260, 0.4332, 0.4438, 0.4470, 0.4517, 0.4530, 0.4553, 0.4564, 0.4576, 0.4612, 0.4675, 0.4694, 0.4741, 0.4752, 0.4990, 0.5113, 0.5140, 0.5150 , 0.5232, 0.5285, 0.5349, 0.5350, 0.5394, 0.5447, 0.5481, 0.5483, 0.5529, 0.5553, 0.5627, 0.5629, 0.5707, 0.5744, 0.5770, 0.5853, 0.5878, 0.5912, 0.5941, 0.6012, 0.6058, 0.6114, 0.6174, 0.6196, 0.6220, 0.6465, 0.6488, 0.6707, 0.6750, 0.6768, 0.6789, 0.6844, 0.6860, 0.6891, 0.6907, 0.6927, 0.7131, 0.7261, 0.7290, 0.7471, 0.7629, 0.7687

Table 7. *Parameter estimates for the milk production data.*

		scheme I		scheme II		scheme III	
Time	**Method**	α	λ	α	λ	α	λ
T_1	MLE	3.4712	2.2533	2.6887	1.9786	3.7154	2.4062
	MPS	3.0965	2.1063	2.3880	1.8480	3.3155	2.2580
T_2	MLE	4.3852	2.3475	3.1446	2.1108	3.0561	2.2409
	MPS	3.8433	2.1897	2.8856	2.0016	2.7934	2.1160

Parametric bootstrap percentile method is used to compute the bootstrap estimates (BootEst) and their corresponding standard error (StdErr). A 95% confidence interval is calculated and the average length of the confidence interval is reported. The output of the parameter estimates along with the bootstrap analysis are summarized in Tables 7 & 8. Results from these Tables reveal that the estimates based on the the MLE and the MPS and based on Schemes I & III, are the closest to those of the complete data set. It is of great importance to notice through this analysis that *MPS*-based estimates consistently outperform those based on the MLE. Their standard errors are the lowest and their confidence intervals are the shortest. This result confirms that the MPS is more efficient than the MLE for estimating the parameters of the Kumaraswamy distribution using Adaptive-IIPH censored data.

7. Conclusions and Recommendations

In this article, the authors considered the Adaptive Type-II Progressive Hybrid censoring data from the Kumaraswamy distribution. This type of censoring satisfies the experiment time limitation and can also be used to simulate different practical situations.

Two types of inference procedures are considered; the MLE and the MPS to estimate the unknown parameters of the Kumaraswamy distribution. It is clear from the simulation that Scheme I is superior to the other schemes as it provides the smallest $Bias$ and the smallest MSE values. It is also observed that the estimates under the MPS outperforms all MLE estimates. On the whole, the MPS estimates are recommended for estimating the shape parameters of the Kumaraswamy distribution based on Adaptive Type-II Progressive Hybrid censoring and based on Scheme I.

Table 8. *Parameter estimates, standard error and length of a 95confidence interval based on bootstrap resamples from the milk production data.*

$T_1 = X_{\frac{4\times m}{5}}$		α			λ		
Scheme	**Method of estimation**	**BootEst. estimates**	**StdError**	**L. B. CI.**	**BootEst. estimates**	**StdError**	**L. B. CI.**
I	MLE	3.1755	0.7071	2.4531	2.2788	0.3007	1.0511
	MPS	2.3422	0.5881	2.4215	2.3482	0.1103	0.4466
II	MLE	3.3527	0.9642	3.9716	2.1014	0.3221	1.2756
	MPS	3.1097	0.8963	3.0263	2.3737	0.2547	0.9409
III	MLE	3.6052	0.9850	3.8260	2.3474	0.3696	1.4476
	MPS	3.4052	0.9721	3.1838	2.7149	0.3532	1.1958

$T_2 = X_m + 2$		α			λ		
Scheme	**Method of estimation**	**BootEst. estimates**	**StdError**	**L. B. CI.**	**BootEst. estimates**	**StdError**	**L. B. CI.**
I	MLE	4.3005	1.3857	5.5748	2.4281	0.3812	1.5022
	MPS	3.2202	0.7349	2.8830	2.1681	0.1633	0.7020
II	MLE	5.6185	1.8901	7.5962	2.5649	0.4183	1.7022
	MPS	3.4011	0.5839	1.8184	2.2692	0.0833	0.2951
III	MLE	4.2498	1.2031	4.5900	2.5923	0.4062	1.6093
	MPS	2.7224	0.5040	1.9385	2.3862	0.0619	0.2312

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