

Uniform Convergence of the Series Expansion of the Multifractional Brownian Motion

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Abstract: In this paper we define the multifractional Brownian motion and we give some properties. we study the uniform Convergence of the Serie expansion. After having determined the covariance function, we give in proposition 2 another proof of almost sure uniform convergence on compact K of the series. We will finish by showing that the m.B.f is locally asymptotically self-similar, with field or fractional Brownian field with Hurst exposant H . One of the problem, for application of multifractional Brownian motion, is the regularity of the function . In the filtered white noise model the increments are no more homogeneous as in fractional Brownian field case. It is obvious when we consider the tangent field associated with a function. Still the multifractional function in the previous model is constant and it is not convient for many applications. We show the uniform convergence of the series on K . We deduce from the previous questions the almost sure uniform convergence of the series to a mBm.

Keywords: Multifractional Brownian Motion, Uniform Convergence, Series Expansion

1. Introduction

The multifractional Brownian motion (mBf) are Gaussian processes which exhibit local-similarity. They have been introduced in the seminal papers of R.F Peltier and Levy-Vehel, Benassi and as extensions to the class of fractional Brownian motion (fBm) processes [14].

This natural extension of fBm results in some sense in a loss of properties the increments of mBm indeed are non stationary and the process is no more self-similar.

One generalize the definition of the fractional Brownian motion of exponent H to the case where H is no longer a constant, but a function of the time index of the processes.

The new process does provide a useful model for host of continuous and non stationary natural signals. It is well known that fBm's are self-similar processes allowing to conveniently describe irregular signals which arise in many situations [7], [4] and [12].

However, the pointwise irregularity of an fBm is the same all along its path.

This last property is sometimes undesirable, since it restricts the fields of application. For instance, fBm have frequently

been used for synthesizing artificial mountains.

Such a modeling assumes that the irregularity of a mountain is everywhere the same. It appears that in reality this assumption is too strong because in particular, one does not take into account erosion phenomena.

Consequently, it should be convenient to relax the constraint of stationarity and be instead able to control the local irregularity.

The mBm seems to be the simplest generalisation of fBm that fulfills this requirement.

Let us give two examples. The long term correlation of the increments of fBm decay as K^{2H-2} , resulting in long range dependence when $H > \frac{1}{2}$ and anti-persistent behavior when $H < \frac{1}{2}$. Also, almost surely, for each its pointwise Hölder exponent is equal to H . Since H rules both ends of the Fourier spectrum, i.e the high frequencies related to the Hölder regularity and the low frequencies related to the long term dependence structure, it is not possible to have at the same time a very irregular local behavior and long range dependence (implying $H > \frac{1}{2}$).

As a consequence, fBm is not adapted to model phenomena which display both these features, such as Internet traffic or

certain highly textured images with strong global organisation. Another example is in the field of image synthesis: fBm has frequently been used for generating artificial mountains. Such a modeling assumes that the regularity of the mountain is everywhere the same. This is not realistic, since it does not take into account erosion or other meteorological phenomena which smooth some parts of mountains more than others.

Multifractional Brownian motion was introduced to overcome there limitations. The basic idea is to replace the real H by a function $h(t)$.

$$W_{H_t} = \frac{1}{\Gamma(H_t + \frac{1}{2})} \left\{ \int_{-\infty}^0 \left[(t-s)^{H_t-\frac{1}{2}} - (-s)^{H_t-\frac{1}{2}} \right] dW(s) + \int_0^t (t-s)^{H_t-\frac{1}{2}} dW(s) \right\}$$

where W denotes the ordinary Brownian motion and the integration is taken in the mean square sense.

The multifractional Brownian motion processes (MBM) are obtained by allowing the self-similarity parameter H of to depend on time.

This can be done by using various representations of the FBM processes.

R.F Peltier and Levy-Vehel [14] have introduced the MBM process

$$Y_{(0,1)}(t) = \int_{\mathbb{R}} \left((t-u)^{H(t)-\frac{1}{2}} - (-u)^{H(t)-\frac{1}{2}} W(du) \right) \quad (2)$$

where

$$x_t^\delta = x^\delta, \quad x > 0 \quad \text{and} \quad x_t^\delta = 0, \quad x \leq 0.$$

Here the function $H(t)$ takes values in the range $(0, 1)$ and $W(du)$ denotes an independently scattered standard Gaussian measure on \mathbb{R} .

The above integral is well-defined in the \mathbb{L}^2 -sense since $0 < H(t) < 1$ and it can be viewed as an integral moving average type representation of the process $Y_{0,1}$. [7] and [11]

On the other hand, $\tilde{Y} = \left\{ \tilde{Y}(t) \right\}_{t \in \mathbb{R}}$ based on a harmonizable integral representation of the fractional Brownian motion. Namely

$$\tilde{Y}(t) = \int_{\mathbb{R}} \left(\frac{e^{it\xi} - 1}{|\xi|^{H(t)+\frac{1}{2}}} \right) \tilde{W}(d\xi) \quad \text{where} \quad H(t) \in (0, 1) \quad (3)$$

and where $\tilde{W}(d\xi)$ denotes a complex-valued Gaussian measure.

Observe that when $H(t) = H = \text{const}$ both $Y_{(0,1)}$ and \tilde{Y} become fractional Brownian motion processes with self-similarity parameter H . When $H(t)$ is non constant but regular, then we expect that continuity, the finite-dimensional distribution of the MBM processes $Y_{(0,1)}(t)$ and $\tilde{Y}(t)$ to be close to those of a FBM process $B_{H(t_0)}(t)$, in a neighborhood of $t_0 \in \mathbb{R}$.

Indeed, Benassi and al have [1] shown, under some regularity conditions on the function $H(t)$, that $\tilde{Y}(t)$ is locally asymptotically self-similar and that its tangent process is the

2. Preliminaries

Definition 2.1. Let $(X, d_X), (Y, d_Y)$ be two metric spaces. A function $f : X \mapsto Y$ is called a Hölder function of exponent $\beta > 0$, if for each $x, y \in X$ such that $d_X(x, y) < 1$ we have

$$d_Y(f(x), f(y)) \leq C d_X(x, y)^\beta. \quad (1)$$

Definition 2.2. Let $H : [0, +\infty) \mapsto (0, 1)$ be a Hölder function of exponent $\beta > 0$. For $t \geq 0$ the following random function, denoted by $W_H(t)$ or $\tilde{W}_H(t)$ is called reduced multifractional Brownian motion with functional parameter H :

FBM process. For results about $Y_{(0,1)}$ see for example, R.F Peltier and Levy-Vehel [14] and also Stoev, Taqqu and Al [5] for results about Gaussian and stable infinite variance processes.

R.F Peltier, Levy-Vehel and Benassi [14] have also shown that, under regularity conditions on the functions $H(t)$, the paths of $Y_{(0,1)}$ and \tilde{Y} have pointwise Hölder exponent at t , equal almost surely to $H(t)$.

The processe $Y_{(0,1)}$ and \tilde{Y} share many properties and it is important to establish their precise connection [8].

Serge Cohen [15] has shown the process \tilde{Y} has the following time domain integral representation:

$$\left\{ \tilde{Y}(t) \right\}_{t \in \mathbb{R}} = \left\{ K_H(t) \int_{\mathbb{R}} \left(|t-u|^{H(t)-\frac{1}{2}} - 1 - |u|^{H(t)-\frac{1}{2}} \right) W(du) \right\} \quad (4)$$

where $K_H(t)$ is a determinisyc function and where the Kernel is

$$\ln \left(\frac{1}{|t-u|} \right) - \ln \left(\frac{1}{|-u|} \right) \quad u \neq 0 \quad u \neq t \quad \text{if} \quad H(t) = \frac{1}{2}.$$

Observe that the stochastic integrals in relations (2) and (4) involve different Kernel functions.

Here, we study the correlation structures of the MBM processes $Y_{(0,1)}$ and \tilde{Y} , extending results of Serge Cohen and Ayache [14].

In fact Stillian and Taqqu [5] shown that the MBM processes $Y_{(0,1)}$ and \tilde{Y} have generally different correlation structures, when the function $H(t)$ is non constant.

Definition 2.3. Let $h : \mathbb{R}^d \rightarrow (0, 1)$ be a measuarable function. A real valued field is called a multifractional Brownian field with multifractional function h , if it admits the harmonisable representation

$$B_h(t) = \frac{1}{(C(h(x)))^{\frac{1}{2}}} \int_{\mathbb{R}^d} \frac{e^{-ix\xi} - 1}{\|\xi\|^{\frac{d}{2}+h(x)}} \tilde{W}(d\xi) \quad (5)$$

where the normalization function is

$$C(s) = \int_{\mathbb{R}^d} \frac{2(1 - \cos(\xi_1))}{\|\xi\|^{d+2s}} \frac{d\xi}{(2\pi)^{\frac{d}{2}}} = \frac{\pi^{\frac{1}{2}} \Gamma(s + \frac{1}{2})}{2^{\frac{d}{2}} \Gamma(2s) \sin(\pi s) \Gamma(s + \frac{d}{2})}$$

where ξ_1 is the first coordinate of the d -dimensional vector ξ . If $d = 1$ multifractional Brownian fields (mBf) are often called multifractional Brownian motion

3. Some Fundamental Results

Proposition 3.1. Let B_h be an mBm with multifractional function h . then

$$E(B_h(x)B_h(y)) = D(h(x), h(y)) \left(\|x\|^{h(x)+h(y)} + \|y\|^{h(x)+h(y)} - \|x-y\|^{h(x)+h(y)} \right) \quad (6)$$

where

$$D(s, t) = \frac{\sqrt{\Gamma(2s+1)\Gamma(2t+1)\sin(\pi s)\sin(\pi t)\Gamma\left(\frac{s+t+1}{2}\right)}}{2\Gamma(s+t+1)\sin\left(\frac{\pi(s+t)}{2}\right)\Gamma\left(\frac{s+t+d}{2}\right)} \quad s, t \in (0, 1)$$

Proof.

$$\begin{aligned} E(B_h(x)B_h(y)) &= E\left(\frac{1}{C(h(x))^{\frac{1}{2}}} \int_{\mathbb{R}^d} \frac{e^{-ix\xi} - 1}{\|\xi\|^{\frac{d}{2}+h(x)}} \tilde{W}(d\xi) \frac{1}{C(h(y))^{\frac{1}{2}}} \int_{\mathbb{R}^d} \frac{e^{iy\xi} - 1}{\|\xi\|^{\frac{d}{2}+h(y)}} \tilde{W}(d\xi)\right) \\ &= \frac{1}{C(h(x))^{\frac{1}{2}} C(h(y))^{\frac{1}{2}}} \int_{\mathbb{R}^d} \frac{(e^{-ix\xi} - 1)(e^{iy\xi} - 1)}{\|\xi\|^{h(x)+h(y)+d}} \frac{d\xi}{(2\pi)^{\frac{d}{2}}} \end{aligned}$$

Fix $x, y \in \mathbb{R}^d$, and let B_H be a standard fractional Brownian field with fractional parameter $H = \frac{h(x)+h(y)}{2}$ because of the harmonizable representation of fractional Brownian field we know

$$\begin{aligned} E(B_h(x)B_h(y)) &= \frac{1}{C_H} \int_{\mathbb{R}^d} \frac{(e^{-ix\xi} - 1)(e^{iy\xi} - 1)}{\|\xi\|^{2H+d}} \frac{d\xi}{(2\pi)^{\frac{d}{2}}} \\ &= \frac{1}{2} (\|x\|^{2H} + \|y\|^{2H} - \|x-y\|^{2H}) \int_{\mathbb{R}^d} \frac{(e^{-ix\xi} - 1)(e^{iy\xi} - 1)}{\|\xi\|^{h(x)+h(y)+d}} \frac{d\xi}{(2\pi)^{\frac{d}{2}}} \\ &= \frac{C_{\frac{h(x)+h(y)}{2}}}{2} \left(\|x\|^{h(x)+h(y)} + \|y\|^{h(x)+h(y)} - \|x-y\|^{h(x)+h(y)} \right) \end{aligned}$$

Replacing by $\frac{h(x)+h(y)}{2}$ in $C(s)$ and using the identity $x\Gamma(x) = \Gamma(x+1)$ yields the announced equality.[2]

Let us define

$$\chi_\lambda(x, y) = \int_{\mathbb{R}} \frac{e^{-ixy} - 1}{|\xi|^{y+\frac{1}{2}}} \overline{\Psi_\lambda(\xi)} \frac{d\xi}{(2\pi)^{\frac{1}{2}}}$$

In the proposition, we give another proof of the almost sure uniform convergence on a compact K of the series

$$= \frac{1}{C(h(x))^{\frac{1}{2}}} \sum_{\lambda \in \Lambda^+} \chi_\lambda(t, h(t)) \eta_\lambda \quad (5) \quad \text{with} \quad \eta_\lambda = H$$

Proposition 3.2. Let h be function locally Hölder continuous with exponent β then we show the following results.

1- the uniform convergence of the series on a compact K

$$\sum_{j \geq J, k \in \mathbb{Z}} \chi_{j,k,1}(t, h(t)) \eta_{j,k,1} \longrightarrow 0 \quad \text{when} \quad J \rightarrow +\infty.$$

2- For $j > 0$ the uniform convergence of the series on K

$$\sum_{|k| \geq K} \chi_{j,k,1}(t, h(t)) \eta_{j,k,1} \longrightarrow 0 \quad \text{when} \quad K \rightarrow +\infty.$$

3- The uniform convergence of the series on K

$$\sum_{j \geq J, k \in \mathbb{Z}} \tilde{\chi}_{0,k,0}(t, h(t)) \eta_{0,k,0} \longrightarrow 0 \quad \text{when} \quad K \rightarrow +\infty.$$

4- The uniform convergence of the series on K

$$\sum_{j \geq J, k \in \mathbb{Z}} \chi_{0,k,0}(t, h(t)) \eta_{0,k,0} \rightarrow 0 \quad \text{when } K \rightarrow +\infty.$$

5- Deduce from the previous questions the almost sure uniform convergence of the series (5) to a mBm.

1- Since h is continuous on K the minimum $a = \inf \{h(t), t \in K\}$ is positive.

$$\left| \sum_{j \geq J, k \in \mathbb{Z}} \chi_{j,k,1}(t, h(t)) \eta_{j,k,1} \right| \leq \sum_{j \geq J} 2^{-aj} \sum_{k \in \mathbb{Z}} \frac{\ln(1+j)^{\frac{1}{2}} + \ln(1+|k|)^{\frac{1}{2}}}{1+|2^j t - k|^2} + \frac{\ln(1+j)^{\frac{1}{2}} + \ln(1+|k|)^{\frac{1}{2}}}{1+|k|^2}$$

and we get

$$\left| \sum_{j \geq J, k \in \mathbb{Z}} \chi_{j,k,1}(t, h(t)) \eta_{j,k,1} \right| \leq C 2^{-a'j}$$

where $a' < a$. It shows the almost sure uniform convergence on K.

2- As in the previous question

$$\left| \sum_{|k| \geq K'} \chi_{j,k,1}(t, h(t)) \eta_{j,k,1} \right| \leq C 2^{-aj} \leq \sum_{|k| \geq K'} \frac{\ln(1+j)^{\frac{1}{2}} + \ln(1+|k|)^{\frac{1}{2}}}{1+|2^j t - k|^2} + \frac{\ln(1+j)^{\frac{1}{2}} + \ln(1+|k|)^{\frac{1}{2}}}{1+|k|^2}$$

Let us show the almost sure uniform convergence on K of $\sum_{|k| \geq K'} \frac{1}{1+|2^j t - k|^2}$. Let us denote by

$$k'(K') = \inf \left\{ |2^j t - k|^2, \quad \forall t \in K, \quad \forall k \in \mathbb{Z}, \quad |k| \geq K' \right\}$$

and let us remark that $\lim_{K' \rightarrow \infty} k'(K') = \infty$ since K is compact, Hence

$$\sum_{|k| \geq K'} \frac{1}{1+|2^j t - k|^2} \leq C \sum_{|k| \geq k'(K')} \frac{1}{k^2} \rightarrow 0 \quad \text{when } K' \rightarrow +\infty$$

The uniform convergence on K of $\sum_{|k| \geq K} \chi_{j,k,1}(t, h(t)) \eta_{j,k,1}$ is a consequence of the previous fact.

3-

$$\left| \sum_{|k| \geq K'} \tilde{\chi}_{0,k,0}(t, h(t)) \eta_{0,k,0} \right| \leq C \sum_{|k| \geq K'} \left(\frac{1}{1+|t-k|^2} + \frac{1}{1+|k|^2} \right) \rightarrow |\eta_{0,k,0}|$$

which almost surely uniformly convergence on K by similar arguments as the one we used in the previous question.

4- Let us consider

$$M_{K'}(t) = \sum_{0 < k < K'} \tilde{\chi}_{0,k,0}(t, h(t)) - \chi_{0,k,0}(t, h(t)) \eta_{0,k,0}$$

$$M_{K'}(t) = E \left(C \int_{|\xi| \leq \frac{4}{3}\pi} \frac{it\xi - \frac{1}{2t^2}\xi^2}{|\xi|^{\frac{1}{2}+h(t)}} \tilde{W}^+(d\xi) / \tau_{K'} \right) \quad \text{where } \tau_{K'} = \sigma(\eta_{0,k,0}, 0 \leq k \leq K')$$

Moreover the process

$$\chi(t, h(t)) = \int_{|\xi| \leq \frac{4}{3}\pi} \frac{it\xi - \frac{1}{2t^2}\xi^2}{|\xi|^{\frac{1}{2}+h(t)}} \tilde{W}^+(d\xi) \quad \text{is almost surely continuous.}$$

M_K converges almost surely uniformly on every compact. The same property is obviously true for

$$\sum_{0 < k < K'} \left(\tilde{\chi}_{0,k,0}(t, h(t)) - \chi_{0,k,0}(t, h(t)) \eta_{0,k,0} \right)$$

Hence we get the result.

5- Since the normalization function is analytic and non vanishing, and h is continuous $\frac{1}{C(h(x))^{\frac{1}{2}}}$ is bounded on K, and the result is a consequence of 1, 2 and 4.

Proposition 3.3. Let $h : \mathbb{R}^d \rightarrow (0, 1)$ be a β -Hölder continuous multifractional function and B_h be the corresponding multifractional Brownian field let us assume $\beta > \sup_{x \in \mathbb{R}^d} h(x)$, then, for every $x \in \mathbb{R}^d$ the mBf is strongly locally asymptotically self-similar, with tangent field a fractional Brownian field with Hurst exponent $H = h(x)$. More precisely

$$\lim_{\varepsilon \rightarrow 0^+} \left(\frac{B_h(x + \varepsilon u) - B_h(x)}{2^{h(x)}} \right) = {}^d B_H(u)_{u \in \mathbb{R}^d} \quad (7)$$

Proof. Let us fix $x \in \mathbb{R}^d$, for every $u_1, u_2 \in \mathbb{R}^d$

$$\begin{aligned} \frac{E(B_h(x + \varepsilon u_1) - B_h(x))(B_h(x + \varepsilon u_2) - B_h(x))}{\varepsilon^{2h(x)}} &= \frac{1}{2\varepsilon^{2h(x)}} \times E(B_h(x + \varepsilon u_1) - B_h(x))^2 \\ &\quad + E(B_h(x + \varepsilon u_2) - B_h(x))^2 \\ &\quad - E(B_h(x + \varepsilon u_2) - B_h(x + \varepsilon u_1))^2 \end{aligned}$$

Hence,

$$\left(\lim_{\varepsilon \rightarrow 0^+} \frac{E(B_h(x + \varepsilon u_1) - B_h(x))(B_h(x + \varepsilon u_2) - B_h(x))}{\varepsilon^{2h(x)}} \right) = \frac{1}{2} \left(\|u_1\|^{2h(x)} + \|u_2\|^{2h(x)} - \|u_1 - u_2\|^{2h(x)} \right)$$

which yields the convergence of finite dimensional margins then, to have the convergence in distribution in the space of continuous functions endowed with the topology of the uniform convergence on every compact, we have to prove the lightness of the renormalized increments.

$$Y_\varepsilon(u) = \frac{B_h(x + \varepsilon u) - B_h(x)}{\varepsilon^{h(x)}}$$

Let us fix a compact K then

$$\lim_{\varepsilon \rightarrow 0} \frac{E(Y_\varepsilon(u_1) - Y_\varepsilon(u_2))^2}{\|u_1 - u_2\|^{2h(x)}} = 1$$

Uniformly when u_1, u_2 are in the compact K , hence

$$\sup_{\varepsilon > 0} E(Y_\varepsilon(u_1) - Y_\varepsilon(u_2))^2 \leq C \|u_1 - u_2\|^{2h(x)}$$

then, one can choose $p \in \mathbb{N}$ such that $ph(x) > d$ since

$$E(Y_\varepsilon(u_1) - Y_\varepsilon(u_2))^{2p} \leq C \|u_1 - u_2\|^{ph(x)}$$

4. Conclusion and Comments

After having determined the covariance function, we give in proposition 2 another proof of almost sure uniform convergence on compact K of the series.

We will finish by showing that the m.B.f is locally asymptotically self-similar, with field or fractional Brownian field with Hurst exponent $H = h(x)$.

Actually if we consider the series expansion of the mBm and consider only a finite number of terms in

$$\sum_{\lambda \in \Lambda} \chi_\lambda(t, h(t)) \eta_\lambda$$

The sum process cannot be in general more regular than h . On the other hand the Hölder exponent of the sample path of the mBm is related to the behavior of $\chi_\lambda(x, y)$ when the frequency part j of $\lambda = (j, k)$ tends to infinity. So, in

some sense, the obstruction to have an irregular multifractional function is a low frequency problem [3]. These considerations lead to propose a generalized model of the mBm, where the multifractional function $h(t)$ may depend of the frequency and, this becomes a function $(t, \xi) \rightarrow H(t, \xi)$ [10] and [6].

Let us discuss more precisely assumptions required for this function.

Hypothesis A function $H : R \times R \rightarrow [a, b] \subset [0, 1]$ is called a frequency lift of a multifractional function h if

$$h(t) = \lim_{\xi \rightarrow +\infty} H(t, \xi)$$

$$H(t, -\xi) = H(t, \xi)$$

H is twice differentiable with aspect to the variable ξ and satisfies

$$(1) \text{ for } 0 < |\xi| \leq \frac{4\pi}{3} \quad H(t, \xi) = b$$

$$(2) \text{ for } \frac{4\pi}{3} < |\xi|, \exists g \text{ such that } 0 < g < \min(\frac{a}{2}, \frac{1}{4}) \text{ and } \exists \beta \text{ such that } b < \beta \leq 1$$

$$\left| \frac{\partial^k H}{\partial \xi^k}(t, \xi) - \frac{\partial^k H}{\partial \xi^k}(t', \xi) \right| \leq C |\xi|^{g-k} |t - t'|^\beta$$

for $t, t' \in [0, 1]$ and $k = 0, 1, 2$

$$\left| \frac{\partial^k H}{\partial \xi^k}(t, \xi) \right| \leq C |\xi|^{g-k}$$

for $t \in [0, 1]$ and $k = 0, 1, 2$.

These assumptions, that are needed to construct an Hölder continuous model, are quite technical.

If one wishes a model similar to mBm but for an irregular multifractional functions h , we have to construct a frequency lift of the desired multifractional function h [13].

In Hypothesis the cut-off at $\xi = \frac{4\pi}{3}$ is arbitrary and be replaced by any non - negative constant. The inequality $0 < g < \min(\frac{a}{2}, \frac{1}{4})$ has a deeper meaning : it expresses the fact that the Hölder constant of $H(t, \xi)$ cannot grow too fast when $\xi \rightarrow +\infty$. Finally, the inequality $b < \beta$ is similar to the conditions $\beta > \sup_{x \in \mathbb{R}^d} h(x)$ in prop 3.

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