

# Two-scale Finite Element Discretizations for Semilinear Parabolic Equations

Fang Liu

School of Statistics and Mathematics, Central University of Finance and Economics, Beijing, China

## Email address:

fliu@cufe.edu.cn

## To cite this article:

Fang Liu. Two-scale Finite Element Discretizations for Semilinear Parabolic Equations. *Applied and Computational Mathematics*. Vol. 9, No. 6, 2020, pp. 179-186. doi: 10.11648/j.acm.20200906.12

**Received:** February 5, 2020; **Accepted:** September 25, 2020; **Published:** November 16, 2020

---

**Abstract:** In this paper, to reduce the computational cost of solving semilinear parabolic equations on a tensor product domain  $\Omega \subset \mathbb{R}^d$  with  $d = 2$  or  $3$ , some two-scale finite element discretizations are proposed and analyzed. The time derivative in semilinear parabolic equations is approximated by the backward Euler finite difference scheme. The two-scale finite element method is designed for the space discretization. The idea of the two-scale finite element method is based on an understanding of a finite element solution to an elliptic problem on a tensor product domain. The high frequency parts of the finite element solution can be well captured on some univariate fine grids and the low frequency parts can be approximated on a coarse grid. Thus the two-scale finite element approximation is defined as a linear combination of some standard finite element approximations on some univariate fine grids and a coarse grid satisfying  $H = O(h^{1/2})$ , where  $h$  and  $H$  are the fine and coarse mesh widths, respectively. It is shown theoretically and numerically that the backward Euler two-scale finite element solution not only achieves the same order of accuracy in the  $H^1(\Omega)$  norm as the backward Euler standard finite element solution, but also reduces the number of degrees of freedom from  $O(h^{-d} \times \tau^{-1})$  to  $O(h^{-(d+1)/2} \times \tau^{-1})$  where  $\tau$  is the time step. Consequently the backward Euler two-scale finite element method for semilinear parabolic equations is more efficient than the backward Euler standard finite element method.

**Keywords:** Two-scale, Finite Element, Combination, Semilinear Parabolic Equation

---

## 1. Introduction

Consider the following semilinear parabolic equation:

$$\begin{cases} u_t - \Delta u = f(u), & (X, t) \in \Omega \times J, \\ u = 0, & (X, t) \in \partial\Omega \times J, \\ u(0) = u_0, & (X, t) \in \Omega \times \{t = 0\}, \end{cases} \quad (1)$$

where  $\Omega \subset \mathbb{R}^d$  is a tensor-product domain ( $d = 2, 3$ ),  $J = (0, T]$  and  $u_t = \frac{\partial u}{\partial t}$ .  $f(\cdot)$  is twice continuously differentiable and  $u_0$  is a given smooth function.

The mathematical theory of Galerkin finite element methods for parabolic equation has been discussed systematically in [37]. Superclose properties of finite element method for linear parabolic problem were studied in [38]. The hp-version discontinuous Galerkin finite element method was proposed for semilinear parabolic problems in [20]. To reduce computational cost, the two-grid methods with finite

difference and finite volume methods were proposed for nonlinear parabolic problems [6, 7, 11, 28]. Two-grid finite element methods for nonlinear Schrödinger equations were developed in [19, 46]. Two-grid mixed finite element methods for nonlinear parabolic problems were studied in [8, 10]. Superconvergence properties of two-grid finite element methods for semilinear parabolic problems were analyzed in [35, 36]. The error estimates of the two-grid discontinuous Galerkin method for nonlinear parabolic equations were studied in [42].

To reduce computational complexity and storage requirements further, in this paper we propose some two-scale finite element discretizations for the semilinear parabolic problem (1). That is, the time derivative in (1) is approximated by the backward Euler finite difference scheme. The two-scale finite element method is used for the space discretization. The two-scale finite element approximations are constructed by using a linear combination of some standard finite element

approximations on a coarse grid and some univariate fine grids. The main idea of the two-scale discretizations is based on an understanding of the frequency resolution of a finite element solution to an elliptic problem. For a solution to an elliptic problem, high frequency components can be well approximated on a fine grid and low frequency components can be computed on a relatively coarse grid (see, e.g., [1, 39, 40, 41, 44]). Moreover, it is known that for elliptic problems on tensor product domains, some high frequencies involve a tensor product of univariate low frequencies, hence they can be handled numerically by a tensor product of univariate fine and coarse grids [4, 5, 15, 25, 30, 31, 34]. It will be shown that on choosing  $H = O(h^{1/2})$  the backward Euler two-scale finite element method achieves the same order of accuracy as the backward Euler standard finite element method while reducing the number of degrees of freedom from  $O(h^{-d} \times \tau^{-1})$  to  $O(h^{-(d+1)/2} \times \tau^{-1})$  where  $\tau$  is the time step for time discretization of (1).

The two-scale finite element method is related to the multi-level sparse grid method [45]. To reduce computational cost, the sparse grid method was proposed for elliptic boundary value problems in which the multi-level basis was used [4, 5, 14, 16, 17, 25, 33, 34, 43]. The combination technique [12, 15], which can be viewed as a variant of the sparse grid method, has been developed. The two-scale finite element discretization uses the two-level basis instead of the multi-level basis [1, 43]. It has been proposed for linear boundary value and eigenvalue problems [13, 18, 30, 31]. The two-level basis is more flexible than the multi-level basis [25, 30].

The so-called superconvergence technique [13, 25, 30, 31, 34, 47] is used in analysis for the two-scale finite element approximations. It has been applied to obtain asymptotic error expansions of the finite element solutions by Lin et al. [2, 23, 26, 27]. A related method is the so-called splitting extrapolation method [21, 22, 48], which is based on the multi-parameter asymptotic error expansions.

This paper is structured as follows. In Section 2, some basic notation is presented. In Section 3, we present some tensor product operators. A standard Galerkin finite element method with the backward Euler finite difference scheme is described. The related error estimate is proposed. In Section 4, the two-scale finite element method with the backward Euler finite difference scheme for semilinear parabolic problem is presented. The theoretical analysis shows that the two-scale finite element method is more efficient than the standard finite element method. Numerical results to illustrate our theory are presented in Section 5. Finally some concluding remarks are given in Section 6.

## 2. Preliminaries

Let  $\Omega = (0, 1)^d$  ( $d = 2, 3$ ) and  $\varpi$  be any measurable subset of  $\Omega^1$ . We will use standard notations for the Sobolev

spaces  $W^{s,p}(\varpi)$  and their associated norms and seminorms [9]. When  $p = 2$ , let  $H^s(\varpi) = W^{s,2}(\varpi)$  and  $\|\cdot\|_{s,\varpi} = \|\cdot\|_{s,2,\varpi}$ . Let  $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$ , where  $v|_{\partial\Omega} = 0$  in the sense of traces. Let  $H^{-1}(\Omega)$  be the dual space of  $H_0^1(\Omega)$ . Write  $(\cdot, \cdot)$  for the  $L^2(\Omega)$  inner product.

Write  $\mathbb{N}_0$  for the set of all nonnegative integers. Let  $\mathbb{Z}_d = \{1, 2, \dots, d\}$ . For each function  $w \in W^{s,p}(\Omega)$ , set

$$(D^\alpha w)(x) = \left( \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \right) (x),$$

where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$  and  $x = (x_1, x_2, \dots, x_d) \in \Omega$ . For general  $d$ -dimensional vector  $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ , let  $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$  and  $\mathbf{x}\alpha = (x_1\alpha_1, \dots, x_d\alpha_d)$ .

For a real number  $t \in \mathbb{R}$ , let

$$t\alpha = (t\alpha_1, \dots, t\alpha_d).$$

The notation  $\alpha \leq \beta$  denotes  $\alpha_i \leq \beta_i$  for  $i \in \mathbb{Z}_d$ ,  $\alpha, \beta \in \mathbb{N}_0^d$  throughout this paper.

We need the so-called mixed Sobolev space:

$$W^{G,3}(\omega) := \{v \in H^2(\omega) : D^\alpha v \in L^2(\omega), \mathbf{0} \leq \alpha \leq 2\mathbf{e}, |\alpha| = 3\},$$

with its natural norms  $\|\cdot\|_{W^{G,3}(\omega)}$  [34], where  $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$ ,  $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^d$  and  $|\alpha| = \alpha_1 + \dots + \alpha_d$ . Let  $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^d$  for each  $i \in \mathbb{Z}_d$ , which means only the  $i^{\text{th}}$  component of  $\mathbf{e}_i$  is 1 and all other ones are 0. Set  $\hat{\mathbf{e}}_i = \mathbf{e} - \mathbf{e}_i$ . The notation  $A \lesssim B$  means  $A \leq CB$  for constant  $C$  which only depends on the data of the problem and does not depend on mesh parameters. We use the notation  $A = \mathcal{O}(B)$  to denote  $|A| \lesssim B$ .

## 3. A Galerkin Finite Element Method

The variational form of (1) is to find  $u : J \mapsto H_0^1(\Omega)$  such that

$$\begin{cases} (u_t, v) + (\nabla u, \nabla v) = (f(u), v), & \forall v \in H_0^1(\Omega), \\ u(X, 0) = u_0(X), & X \in \Omega, \end{cases} \quad (2)$$

where  $(f(u), v) = \int_\Omega f v$ .

Let  $T^h[0, 1]$  be the uniform mesh, where  $h = 1/N \in [0, 1]$  is the mesh size with  $N \in \mathbb{N}_0 \setminus \{0\}$ . Let  $S^h[0, 1] \subset C[0, 1]$  be the associated piecewise linear finite element space.

Set  $\mathbf{h} = (h_1, \dots, h_d)$  and  $h = \max_{1 \leq j \leq d} h_j$  with  $h_j = 1/N_j$  for  $N_j \in \mathbb{N}_0 \setminus \{0\}$ ,  $j \in \mathbb{Z}_d$ . Define a tensor-product mesh on  $\bar{\Omega} = [0, 1]^d$  by

$$T^{\mathbf{h}}(\Omega) := T^{h_1}[0, 1] \times \cdots \times T^{h_d}[0, 1].$$

<sup>1</sup>Though our analysis is for  $\Omega$ , it can be generalized to any tensor product domain.

The corresponding tensor product spaces of piecewise  $d$ -linear functions on  $\Omega$  are

$$S^h(\Omega) := S^{h_1}[0, 1] \otimes \cdots \otimes S^{h_d}[0, 1]$$

and

$$S_0^h(\Omega) := S^h(\Omega) \cap H_0^1(\Omega).$$

The standard Lagrange interpolation operator  $I_h : C(\bar{\Omega}) \rightarrow S^h(\Omega)$  is defined by

$$I_h := I_{h_1 e_1} \circ \cdots \circ I_{h_d e_d} \equiv \prod_{i=1}^d I_{h_i e_i}.$$

Here,  $I : C(\bar{\Omega}) \rightarrow C(\bar{\Omega})$  denotes the identity operator. The Ritz projector  $R_h : H_0^1(\Omega) \mapsto S_0^h(\Omega)$  is defined by

$$(\nabla(w - R_h w), \nabla v) = 0, \quad \forall v \in S_0^h(\Omega), \quad (3)$$

for each  $w \in H_0^1(\Omega)$ . There holds that [3, 9, 23, 25, 48]

$$\|w - R_h w\|_{0,\Omega} + h\|w - R_h w\|_{1,\Omega} \lesssim h^2 \|w\|_{2,\Omega}, \quad (4)$$

$$\|I_h w - R_h w\|_{1,\Omega} \lesssim h^2 \|w\|_{W^{G,3}(\Omega)}, \quad (5)$$

for each  $w \in H_0^1(\Omega) \cap W^{G,3}(\Omega)$ .

Let  $\{t_n | t_n = n\tau, 0 \leq n \leq N\}$  be a uniform partition in time with time step  $\tau$  and  $u^n = u(X, t_n)$ . For a sequence of a functions  $\{\phi^n\}_{n=0}^N$ , we denote  $D_\tau \phi^n = (\phi^n - \phi^{n-1})/\tau$ . Then with the mesh and trial space in Section 2, the backward Euler scheme of (2) is: find  $u_h^n \in S_0^h(\Omega)$  for  $n = 1, 2, \dots, N$  such that

$$\begin{cases} (D_\tau u_h^n, v) + (\nabla u_h^n, \nabla v) = (f(u_h^n), v_h) & \forall v \in S_0^h(\Omega), \\ u_h^0 = R_h u_0(X), & X \in \Omega. \end{cases} \quad (6)$$

The following Lemma for superclose property is a generalization of Theorem 2.1 in [36] where only two-dimensional case is mentioned and  $H^3(\Omega)$  instead of  $W^{G,3}(\Omega)$  is used.

**Lemma 3.1.** Let  $u$  and  $u_h^n$  be the solution of (2) and (6), respectively. Assume that  $u \in L^\infty(J; W^{G,3}(\Omega))$ ,  $u_t \in L^2(J; H^2(\Omega))$ , and  $u_{tt} \in L^\infty(J; L^2(\Omega))$ , then

$$\begin{aligned} \|u_h^n - I_h u^n\|_{1,\Omega} &\leq Ch^2 (\|u\|_{L^\infty(J; W^{G,3}(\Omega))}^2 \\ &+ \|u_t\|_{L^2(J; H^2(\Omega))})^{1/2} + C\tau \|u_{tt}\|_{L^\infty(J; L^2(\Omega))}. \end{aligned} \quad (7)$$

**Algorithm 4.1.** For  $n = 1, 2, \dots, N$ ,

1. find  $u_{H_e}^n \in S_0^{He}(\Omega)$  such that

$$\begin{cases} (D_\tau u_{H_e}^n, v) + (\nabla u_{H_e}^n, \nabla v) = (f(u_{H_e}^n), v) & \forall v \in S_0^{He}(\Omega), \\ u_{H_e}^0 = R_{H_e} u_0, \end{cases}$$

and for each  $i \in \mathbb{Z}_d$ , find  $u_{h e_i + H \hat{e}_i}^n \in S_0^{h e_i + H \hat{e}_i}(\Omega)$  such that

$$\begin{cases} (D_\tau u_{h e_i + H \hat{e}_i}^n, v) + (\nabla u_{h e_i + H \hat{e}_i}^n, \nabla v) = (f(u_{h e_i + H \hat{e}_i}^n), v) & \forall v \in S_0^{h e_i + H \hat{e}_i}(\Omega), \\ u_{h e_i + H \hat{e}_i}^0 = R_{h e_i + H \hat{e}_i} u_0. \end{cases}$$

*Proof.* By (5) and the proof of Theorem 2.1 in [36], we can obtain this conclusion directly.

By Lemma 3.1 we have the following result immediately. Similar result has been proposed in [35] for two-dimensional case with different proof.

**Theorem 3.1.** Let  $u$  and  $u_h^n$  be the solution of (2) and (6), respectively. Assume that  $u \in L^\infty(J; W^{G,3}(\Omega))$ ,  $u_t \in L^2(J; H^2(\Omega))$ , and  $u_{tt} \in L^\infty(J; L^2(\Omega))$ , then

$$\|u^n - u_h^n\|_{1,\Omega} \lesssim h + \tau. \quad (8)$$

## 4. Two-scale Finite Element Method

In this section, we propose two-scale finite element discretizations for the semilinear parabolic problem.

Let  $h, H \in (0, 1)$  and assume that  $H/h$  is a positive integer. In practice we choose  $h \ll H$ . Let  $w_{h\alpha+H(e-\alpha)} \in S_0^{h\alpha+H(e-\alpha)}(\Omega)$  for  $0 \leq \alpha \leq e$ . Following [29, 30, 32], we define a Boolean sum as follows

$$\begin{aligned} B_{H_e}^h w_{H_e} &= \sum_{i=1}^d w_{h e_i + H \hat{e}_i} - (d-1) w_{H_e}, \\ \forall w_{H_e} &\in S_0^{He}(\Omega), \end{aligned} \quad (9)$$

in which each summand uses a mesh of width  $h$  in at most one coordinate direction. For example, if  $d = 2$ , we have  $B_{H,H}^h w_{h,h} = w_{h,H} + w_{H,h} - w_{H,H}$ . If  $d = 3$ , we have  $B_{H,H,H}^h w_{h,h,h} = w_{h,H,H} + w_{H,h,H} + w_{H,H,h} - 2w_{H,H,H}$ .

In our analysis, we need the following result [13, 22, 24, 29, 30].

**Proposition 4.1.** If  $w \in C(\bar{\Omega}) \cap W^{G,3}(\Omega)$ , then

$$\begin{aligned} H \|B_{H_e}^h I_{h_e} w - I_{H_e} w\|_{1,\Omega} + \|B_{H_e}^h I_{h_e} w \\ - I_{H_e} w\|_{0,\Omega} &\lesssim H^3 \|w\|_{W^{G,3}(\Omega)}. \end{aligned} \quad (10)$$

Our first algorithm is the basic two-scale finite element discretization with the backward Euler finite difference scheme, in which at each time step the two-scale finite element solution is a linear combination of standard finite element solutions on a coarse grid and some univariant fine grids.

## 2. (Two-scale solution) Set

$$(\tilde{u}_{H\mathbf{e}}^h)^n = B_{H\mathbf{e}}^h u_{h\mathbf{e}}^n = \sum_{i=1}^d u_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n - (d-1)u_{H\mathbf{e}}^n.$$

*Theorem 4.1.* Assume that  $u \in H_0^1(\Omega) \cap W^{G,3}(\Omega)$  and  $u$  is the solution of (2), then

$$\|u^n - (\tilde{u}_{H\mathbf{e}}^h)^n\|_{1,\Omega} \lesssim h + H^2 + \tau. \quad (11)$$

*Proof.* Because  $C(\bar{\Omega}) \cap H_0^1(\Omega) \cap W^{G,3}(\Omega)$  is dense in  $H_0^1(\Omega) \cap W^{G,3}(\Omega)$ , we only need to prove (11) for  $u \in C(\bar{\Omega}) \cap H_0^1(\Omega) \cap W^{G,3}(\Omega)$ . Using the definition  $(\tilde{u}_{H\mathbf{e}}^h)^n = B_{H\mathbf{e}}^h u_{h\mathbf{e}}^n$ , we have

$$\begin{aligned} \|u_{h\mathbf{e}}^n - (\tilde{u}_{H\mathbf{e}}^h)^n\|_{1,\Omega} &\leq \|I_{h\mathbf{e}} u^n - u_{h\mathbf{e}}^n\|_{1,\Omega} + \sum_{i=1}^d \|I_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} u^n - u_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n\|_{1,\Omega} \\ &\quad + (d-1) \|I_{H\mathbf{e}} u^n - u_{H\mathbf{e}}^n\|_{1,\Omega} + \|I_{h\mathbf{e}} u^n - B_{H\mathbf{e}}^h I_{h\mathbf{e}} u^n\|_{1,\Omega} \\ &\lesssim H^2 + \tau, \end{aligned}$$

where Lemma 3.1 and Proposition 4.1 are used in the last inequality. Thus, we have

$$\|u^n - (\tilde{u}_{H\mathbf{e}}^h)^n\|_{1,\Omega} \lesssim h + H^2 + \tau.$$

Our second two-scale finite element discretization with the backward Euler finite difference scheme is described in Algorithm 4.2. In this algorithm, at each time step we solve a semilinear system on a coarse grid and some linear systems on some partially fine grids.

*Algorithm 4.2.* For  $n = 1, 2, \dots, N$ ,

1. Compute semilinear problem on the coarse grid: find  $u_{H\mathbf{e}}^n \in S_0^{H\mathbf{e}}(\Omega)$  such that

$$\begin{cases} (D_\tau u_{H\mathbf{e}}^n, v) + (\nabla u_{H\mathbf{e}}^n, \nabla v) = (f(u_{H\mathbf{e}}^n), v) & \forall v \in S_0^{H\mathbf{e}}(\Omega), \\ u_{H\mathbf{e}}^0 = R_{H\mathbf{e}} u_0. \end{cases}$$

2. Compute linear problems on some partially fine grids in parallel: for each  $i \in \mathbb{Z}_d$ , find  $\tilde{u}_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n \in S_0^{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}(\Omega)$  such that

$$\begin{cases} (D_\tau \tilde{u}_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n, v) + (\nabla \tilde{u}_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n, \nabla v) = (f(u_{H\mathbf{e}}^n) + f'(u_{H\mathbf{e}}^n)(\tilde{u}_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n - u_{H\mathbf{e}}^n), v) & \forall v \in S_0^{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}(\Omega), \\ \tilde{u}_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^0 = R_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} u_0. \end{cases}$$

3. (Two-scale solution) Set

$$(\tilde{u}_{H\mathbf{e}}^h)^n = \sum_{i=1}^d \tilde{u}_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n - (d-1)u_{H\mathbf{e}}^n.$$

*Theorem 4.2.* Assume that  $u \in H_0^1(\Omega) \cap W^{G,3}(\Omega)$  and  $u$  is the solution of (2), then

$$\|u^n - (\tilde{u}_{H\mathbf{e}}^h)^n\|_{1,\Omega} \lesssim h + H^2 + \tau. \quad (12)$$

*Proof.* Again we only need to prove (12) for  $u \in C(\bar{\Omega}) \cap H_0^1(\Omega) \cap W^{G,3}(\Omega)$  since  $C(\bar{\Omega}) \cap H_0^1(\Omega) \cap W^{G,3}(\Omega)$  is dense in  $H_0^1(\Omega) \cap W^{G,3}(\Omega)$ . Using the definition of  $(\tilde{u}_{H\mathbf{e}}^h)^n$ , we have

$$\begin{aligned} \|u_{h\mathbf{e}}^n - (\tilde{u}_{H\mathbf{e}}^h)^n\|_{1,\Omega} &= \left\| \sum_{i=1}^d \tilde{u}_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n - (d-1)u_{H\mathbf{e}}^n - u_{h\mathbf{e}}^n \right\|_{1,\Omega} \\ &\leq \sum_{i=1}^d \|\tilde{u}_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n - u_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n\|_{1,\Omega} + \|u_{h\mathbf{e}}^n - B_{H\mathbf{e}}^h u_{h\mathbf{e}}^n\|_{1,\Omega}. \end{aligned}$$

By Theorem 3.1 in [36], which can be generalized directly for three-dimensional case, we have

$$\|\tilde{u}_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n - u_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n\|_{1,\Omega} \lesssim \|\tilde{u}_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n - I_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} u^n\|_{1,\Omega} + \|I_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} u^n - u_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n\|_{1,\Omega} \lesssim H^2 + \tau.$$

Thus, we have

$$\|u^n - (\tilde{u}_{He}^h)^n\|_{1,\Omega} \lesssim h + H^2 + \tau.$$

**Remark 4.1.** Comparing Theorem 3.1 with Theorems 4.1 and 4.2, we can see that if we choose  $h = O(H^2)$  then the approximate solutions  $(\tilde{u}_{He}^h)^n$  and  $(\tilde{u}_{He}^h)^n$ , obtained from Algorithms 4.1 and 4.2, can achieve the same accuracy as  $u_{he}^n$  from the standard discretization (6), while both  $(\tilde{u}_{He}^h)^n$  and  $(\tilde{u}_{He}^h)^n$  require only  $O(h^{-(d+1)/2} \times \tau^{-1})$  degrees of freedom compared with  $O(h^{-d} \times \tau^{-1})$  degrees of freedom required by  $u_{he}^n$ . Hence the two-scale finite element discretizations (Algorithms 4.1 and 4.2) are more efficient than the standard finite element discretization (6). Moreover, in Algorithm 4.2 one only need to solve the semilinear parabolic problem on the coarse mesh, hence Algorithm 4.2 is even better than Algorithm 4.1.

## 5. Numerical Examples

In this section, some numerical examples are presented to show the efficiency of the two-scale finite element method. To optimize the computational cost, in all of the numerical experiments we choose  $h = H^2$  and  $\tau = h$ .

**Example 1.** Consider the following semilinear parabolic

equation:

$$\begin{cases} u_t - \Delta u - \sin u = g, & (X, t) \in \Omega \times J, \\ u = 0, & (X, t) \in \partial\Omega \times J, \\ u(0) = u_0, & (X, t) \in \Omega \times \{t = 0\}, \end{cases} \quad (13)$$

where  $\Omega = (0, 1) \times (0, 1)$  and  $J \in (0, 1]$ .  $g$  and  $u_0$  are computed from the exact solution

$$u(x, y, t) = e^{-t} \sin(\pi x) \sin(\pi y).$$

In Tables 1 and 2, the numerical results at  $t = 0.5$  and  $t = 1.0$  are shown, respectively. The approximate solution  $u_{h,h}^n$  is obtained from the standard finite element discretization (6) while  $(\tilde{u}_{H,H}^h)^n$  and  $(\tilde{u}_{H,H}^h)^n$  are computed by Algorithms 4.1 and 4.2, respectively. These results support Theorems 3.1, 4.1, and 4.2. It can be seen that the two-scale finite element solutions  $(\tilde{u}_{H,H}^h)^n$  and  $(\tilde{u}_{H,H}^h)^n$  achieves the same order of accuracy as the standard finite element solution  $u_{h,h}^n$ , but with much less computational cost. For example, when  $h = 1/100$ , at each time step, the exact degrees of freedom to get  $(\tilde{u}_{H,H}^h)^n$  and  $(\tilde{u}_{H,H}^h)^n$  is only  $100 \times 10$  while that for the standard finite element solution is  $100 \times 100$ . Thus the two-scale finite element discretizations are more efficient than the standard finite element discretization. Moreover, in the process of computing  $(\tilde{u}_{H,H}^h)^n$ , one only need solve semilinear problem on the coarse grid. Hence it is even better than  $(\tilde{u}_{H,H}^h)^n$ .

**Table 1.** The errors at  $t = 0.5$ . Compare the two-scale finite element method with the standard finite element method.

$1/h \times 1/H$	$\ u^n - u_{h,h}^n\ _1$	$\ u^n - (\tilde{u}_{H,H}^h)^n\ _1$	$\ u^n - (\tilde{u}_{H,H}^h)^n\ _1$
$4 \times 2$	0.30343903	0.31174759	0.31173088
$16 \times 4$	0.07635569	0.07704568	0.07704581
$64 \times 8$	0.01910041	0.01914644	0.01914647
$100 \times 10$	0.01222476	0.01224376	0.01224377
convergence rate	$\mathcal{O}(h)$	$\mathcal{O}(h)$	$\mathcal{O}(h)$

**Table 2.** The errors at  $t = 1.0$ . Compare the two-scale finite element method with the standard finite element method.

$1/h \times 1/H$	$\ u^n - u_{h,h}^n\ _1$	$\ u^n - (\tilde{u}_{H,H}^h)^n\ _1$	$\ u^n - (\tilde{u}_{H,H}^h)^n\ _1$
$4 \times 2$	0.18404477	0.18907448	0.18907027
$16 \times 4$	0.04631234	0.04673028	0.04673031
$64 \times 8$	0.01158504	0.01161291	0.01161292
$100 \times 10$	0.00741472	0.00742623	0.00742623
convergence rate	$\mathcal{O}(h)$	$\mathcal{O}(h)$	$\mathcal{O}(h)$

**Example 2.** Consider the following semilinear parabolic equation:

$$\begin{cases} u_t - \Delta u + u^3 = g, & (X, t) \in \Omega \times J, \\ u = 0, & (X, t) \in \partial\Omega \times J, \\ u(0) = u_0, & (X, t) \in \Omega \times \{t = 0\}, \end{cases} \quad (14)$$

where  $\Omega = (0, 1) \times (0, 1)$  and  $J \in (0, 1]$ .  $g$  and  $u_0$  are computed from the exact solution

$$u(x, y, t) = e^{-t+x+y} xy(1-x)(1-y).$$

Tables 3 and 4 show the numerical results at  $t = 0.5$  and  $t = 1.0$ , respectively. The numerical results also support Theorems 3.1, 4.1, and 4.2. It is shown that the two-scale finite element discretizations, that is, Algorithms 4.1 and 4.2, are very efficient compared with the standard finite element discretization (6).

**Table 3.** The errors at  $t = 0.5$ . Compare the two-scale finite element method with the standard finite element method.

$1/h \times 1/H$	$\ u^n - u_{h,h}^n\ _1$	$\ u^n - (\tilde{u}_{H,H}^h)^n\ _1$	$\ u^n - (\tilde{u}_{H,H}^h)^n\ _1$
$4 \times 2$	0.09193256	0.09878091	0.09877565
$16 \times 4$	0.02326492	0.02393244	0.02394007
$64 \times 8$	0.00582061	0.00586443	0.00589146
$100 \times 10$	0.00372531	0.00374325	0.00378199
convergence rate	$\mathcal{O}(h)$	$\mathcal{O}(h)$	$\mathcal{O}(h)$

**Table 4.** The errors at  $t = 1.0$ . Compare the two-scale finite element method with the standard finite element method.

$1/h \times 1/H$	$\ u^n - u_{h,h}^n\ _1$	$\ u^n - (\tilde{u}_{H,H}^h)^n\ _1$	$\ u^n - (\tilde{u}_{H,H}^h)^n\ _1$
$4 \times 2$	0.05576004	0.05991326	0.05991179
$16 \times 4$	0.01411090	0.01451574	0.01451675
$64 \times 8$	0.00353038	0.00355696	0.00356019
$100 \times 10$	0.00225952	0.00227040	0.00227468
convergence rate	$\mathcal{O}(h)$	$\mathcal{O}(h)$	$\mathcal{O}(h)$

## 6. Conclusion

In this paper, the backward Euler two-scale finite element algorithms (Algorithms 4.1 and 4.2) for semilinear parabolic problems are proposed. Theoretical analysis and numerical examples show that on choosing  $h = \mathcal{O}(H^2)$  the backward Euler two-scale finite element approximations yield the same accuracy as the backward Euler standard finite element solution but much less computational cost. Besides, on the univariate fine grids, some semilinear problems are solved in Algorithm 4.1 while some linear problems are solved in Algorithm 4.2. Hence Algorithm 4.2 is even more efficient than Algorithm 4.1.

## Acknowledgements

The author is grateful to Professor Aihui Zhou for his helpful advice on her research and for his valuable comments on the draft of paper. The author also thanks Professor Huajie Chen for his helpful discussion and some codes. This work was supported by the National Natural Science Foundation of China under grant 11771467.

## References

- [1] Bank R E. Hierarchical bases and the finite element method. Acta Numer, Cambridge University Press, Cambridge, UK, 1996, 5: 1-43.
- [2] Blum H, Lin Q, Rannacher R. Asymptotic error expansion and Richardson extrapolation for linear finite elements. Numer Math, 1986, 49: 11-37.
- [3] Brenner S C, Scott L R. The Mathematical Theory of Finite Element Methods, 3rd ed.. Texts in Applied Mathematics, Springer, New York, 2008, 15.
- [4] Bungartz H-J, Griebel M. Sparse grids. Acta Numer, 2004, 13: 1-123.
- [5] Bungartz H-J, Griebel M, R de U. Extrapolation, combination, and sparse grid techniques for elliptic boundary value problems. Comput Methods Appl Mech Engrg, 1994, 116: 243-252.
- [6] Chen C, Yang M, Bi C. Two-grid methods for finite volume element approximations of nonlinear parabolic equations. J Comput Appl Math, 2009, 228: 123-132.
- [7] Chen C, Liu W. Two-grid finite volume element methods for semilinear parabolic problems. Appl Numer Math, 2010, 60 (1): 10-18.
- [8] Chen Y, Chen L, Zhang X. Two-grid method for nonlinear parabolic equations by expanded mixed finite element methods. Numer Methods Partial Differential Equations, 2013, 29 (4): 1238-1256.
- [9] Ciarlet P G. The Finite Element Method for Elliptic Problems. Studies in Mathematics and Its Applications, North-Holland Publishing Co., Amsterdam, 1978, 4.
- [10] Dawson C, Wheeler M. Two-grid methods for mixed finite element approximations of nonlinear parabolic equations. Contemp Math, 1994, 180: 191-203.
- [11] Dawson C, Wheeler M, Woodward C. A two-grid finite difference scheme for nonlinear parabolic equations. SIAM J Numer Anal, 1998, 35 (2): 435-452.
- [12] Delvos F. d-Variate Boolean interpolation. J Approx Theory, 1982, 34: 99-114.
- [13] Gao X, Liu F, Zhou A. Three-scale finite element eigenvalue discretizations. BIT, 2008, 48: 533-562.

- [14] Garcke J, Griebel M. On the computation of the eigenproblems of hydrogen and helium in strong magnetic and electric fields with the sparse grid combination technique. *J Comput Phys*, 2000, 165: 694-716.
- [15] Griebel M, Schneider M, Zenger C. A combination technique for the solution of sparse grid problem. *Iterative Methods in Linear Algebra* (P. de Groen and P. Beauwens, eds). IMACS, North Holland, Amsterdam, 1992, 263-281.
- [16] Hegland M. Adaptive sparse grids. *ANZIAM J*, 2003, 44: C335-C353.
- [17] Hegland M, Garcke J, Challis V. The combination technique and some generalisations. *Linear Algebra Appl*, 2007, 420: 249-275.
- [18] Hennart J P, Mund E H. On the  $h$ - and  $p$ -versions of the extrapolated Gordon's projector with applications to elliptic equations. *SIAM J Sci Statist Comput*, 1988, 9: 773-791.
- [19] Jin J, Wei N, Zhang H. A two-grid finite-element method for the nonlinear Schrödinger equation. *J Comput Math*, 2015, 33 (2): 146-157.
- [20] Lasis A, Süli E. hp-version discontinuous Galerkin finite element method for semilinear parabolic problems. *SIAM J Numer Anal*, 2007, 45 (4): 1544-1569.
- [21] Liao X, Zhou A. A multi-parameter splitting extrapolation and a parallel algorithm for elliptic eigenvalue problem. *J Comput Math*, 1998, 16: 213-220.
- [22] Liem C B, Lü T, Shih T M. The Splitting Extrapolation Method. *Series on Applied Mathematics*, World Scientific Publishing Co., River Edge, NJ, 1995, 7.
- [23] Lin Q, Lin J. *Finite Element Methods: Accuracy and Improvement*. Science Press, Beijing, 2006.
- [24] Lin Q, Lü T. Asymptotic expansions for finite element eigenvalues and finite element solution. *Extrapolation procedures in the finite element method*, Bonner Math Schriften, Univ Bonn, Bonn, 1984, 158: 1-10.
- [25] Lin Q, Yan N, Zhou A. A sparse finite element method with high accuracy. Part I. *Numer Math*, 2001, 88: 731-742.
- [26] Lin Q, Zhu Q. Undirectional extrapolations of finite difference and finite elements. *J Engrg Math*, 1984, 1: 1-12 (in Chinese).
- [27] Lin Q, Zhu Q. *The Preprocessing and Postprocessing for the Finite Element Method*. Shanghai Sci Tech Press, 1994 (in Chinese).
- [28] Liu W, Rui H, Bao Y. Two kinds of two-grid algorithms for finite difference solutions of semilinear parabolic equations. *J Sys Sci Math Sci*, 2010, 30 (2): 181-190.
- [29] Liu F, Stynes M, Zhou A. Postprocessed two-scale finite element discretizations, part I. *SIAM J Numer Anal*, 2011, 49: 1947-1971.
- [30] Liu F, Zhou A. Two-scale finite element discretizations for partial differential equations. *J Comput Math*, 2006, 24: 373-392.
- [31] Liu F, Zhou A. Localizations and parallelizations for two-scale finite element discretizations. *Commun Pure Appl Anal*, 2007, 6: 757-773.
- [32] Liu F, Zhou A. Two-scale Boolean Galerkin discretizations for Fredholm integral equations of the second kind. *SIAM J Numer Anal*, 2007, 45: 296-312.
- [33] Pflaum C. Convergence of the combination technique for second-order elliptic differential equations. *SIAM J Numer Anal*, 1997, 34: 2431-2455.
- [34] Pflaum C, Zhou A. Error analysis of the combination technique. *Numer Math*, 1999, 84: 327-350.
- [35] Shi D, Yang H. Unconditional optimal error estimates of a two-grid method for semilinear parabolic equations. *Appl Math Comput*, 2017, 310: 40-47.
- [36] Shi D, Mu P, Yang H. Superconvergence analysis of a two-grid method for semilinear parabolic equations. *Appl Math Lett*, 2018, 84: 34-41.
- [37] Thomée V. *Galerkin Finite Element Methods for Parabolic Problems*. Springer-Verlag, Berlin, 1984.
- [38] Thomée V, Xu J, Zhang N. Superconvergence of the gradient in piecewise linear finite-element approximation to a parabolic problem. *SIAM J Numer Anal*, 1989, 26 (3): 553-573.
- [39] Xu J. Two-grid discretization techniques for linear and nonlinear PDEs. *SIAM J Numer Anal*, 1996, 33: 1759-1777.
- [40] Xu J, Zhou A. Local and parallel finite element algorithms based on two-grid discretizations. *Math Comp*, 2000, 69: 881-909.
- [41] Xu, J, Zhou, A. A two-grid discretization scheme for eigenvalue problems. *Math Comput*, 2001, 70: 17-25.
- [42] Yang J, Xing X. A two-grid discontinuous Galerkin method for a kind of nonlinear parabolic problems. *Appl Math Comput*, 2019, 346: 96-108.
- [43] Yserentant H. On the multi-level splitting of finite element spaces. *Numer Math*, 1986, 49: 379-412.
- [44] Yserentant, H. *Old and new convergence proofs for multigrid methods*. Acta Numerica, Cambridge University Press, Cambridge, UK, 1993, 285-326.

- [45] Zenger, C. Sparse grids. *Parallel Algorithms for Partial Differential Equations* (Kiel, 1990), *Notes Numer Fluid Mech*, Vieweg, Braunschweig, 1991, 31: 241-251.
- [46] Zhang H, Jin J, Wang J. Two-grid finite-element method for the two-dimensional time-dependent Schrödinger equation. *Adv Appl Math Mech*, 2013, 5(2): 180-193.
- [47] Zhou A, Li J. The full approximation accuracy for the stream function-vorticity-pressure method. *Numer Math*, 1994, 68: 427-435.
- [48] Zhou A, Liem C, Shih T, Lü T. A multi-parameter splitting extrapolation and a parallel algorithm. *Systems Sci Math Sci*, 1997, 10: 253-260.