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Search Directions in Infeasible Newton's Method for Computing Weighted Analytic Center for Linear Matrix Inequalities

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Abstract: Four different search directions for Infeasible Newton's method for computing the weighted analytic center defined by a system of linear matrix inequality constraints are studied. Newton's method is applied to find the weighted analytic center and the starting point can be infeasible, that is, outside the feasible region determined by the linear matrix inequality constraints. More precisely, Newton's method is used to solve system of equations given by the KKT optimality conditions for the weighted analytic center. The search directions for the Newton's method considered are the ZY, ZY+YZ, Z^{-1} and NT methods that have been used in semidefinite programming. Backtracking line search is used for the Newton's method. Numerical experiments are used to compare these search direction methods on randomly generated test problems by looking at the iterations and time taken to compute the weighted analytic center. The starting points are picked randomly outside the feasible region. Our numerical results indicate that ZY+YZ and ZY are the best methods. The ZY+YZ method took the least number of iterations on average while ZY took the least time on average and they handle weights better than the other methods when some of the weights are very large relative to the other weights. These are followed by NT method and then Z^{-1} method.

Keywords: Linear Matrix Inequalities, Weighted Analytic Center, Newton's Method, Semidefinite Programming

1. Introduction

Consider a system of linear matrix inequality (LMI) constraints given below:

$$A^{(j)}(x) := A_0^{(j)} + \sum_{i=1}^{n} x_i A_i^{(j)} \succeq 0, \quad (j = 1, 2, \ldots, q), \tag{1}$$

where $x \in \mathrm{IR}^n$ is a variable and each $A_i^{(j)}$ is a m_j x m_j symmetric matrix for $i=0,1,...,n$. LMI constraints have applications in a variety of areas including engineering, geometry and statistics [1, 9]. Assume that feasible determined by the constraints is bounded and has a nonempty interior.

Let $\mathcal{R}$ denote the feasible region defined by the inequalities (1). Given $\omega > 0$, define the barrier function $\varphi_\omega(x) : \mathrm{IR}^n \longrightarrow \mathrm{IR}$ by:

$$\phi_\omega(x) = \begin{cases} \sum_{j=1}^{q} \omega_j \operatorname{logdet}[(A^{(j)}(x))^{-1}] & \text{if } x \in int(\mathcal{R}) \\ \infty & otherwise \end{cases}$$

The weighted analytic center for the system (1) was introduced by Pressman and Jibrin [7], and discussed in a paper by Jibrin and Swift [5]. It is defined by:

$$x_{ac}(\omega) = argmin\{\varphi_\omega(x) \mid x \in \mathrm{IR}^n\}$$

This is a more general form of the determinant maximization problem considered by Vandenberghe et al. [11]. In the special case of linear constraints, weighted analytic center has been studied extensively in the past [3]. This definition of weighted analytic center for LMIs extends that of linear constraints studied by Atkinson and Vaidya [3].

Jibrin presents Infeasible Newton's method for finding weighted analytic center for the system (1) [4]. A feasible starting point is not required to start the method. In this paper, four search directions for the method, namely: ZY, ZY+YZ, Z^{-1} and NT methods are compared in computing the weighted analytic center. These search directions have been used in the past in the problem of semidefinite programming [2]. The ZY+YZ method took the least number of iterations on average while ZY took the least time on average and they handle weights better than the other methods when some of the weights are very large relative to the other weights. These are followed by NT method and then Z^{-1} method. The results agree with what is known in semidefinite programming, where ZY+YZ and ZY are found to be more efficient among the four methods.

2. Infeasible Newton's Method for Computing Weighted Analytic Center

This section briefly describes infeasible Newton's method for computing weighted analytic center for the system (1) given by Jibrin in an earlier paper [4].

The optimality conditions for computing the weighted analytic center $x_{ac}(\omega)$ are given by:

$$A_0^{(j)} + \sum_{i=1}^{n} x_i A_i^{(j)} - Y^{(j)} = 0, \ (j = 1, \ldots, q) \tag{2}$$

$$\sum_{j=1}^{q} A_i^{(j)} \bullet Z^{(j)} = 0, \ (i = 1, \ldots, n) \tag{3}$$

$$Z^{(j)} Y^{(j)} = \omega_j I_{m_j}, \ (j = 1, \ldots, q) \tag{4}$$

$$Y^{(j)} \succ 0, \ (j = 1, \ldots, q) \tag{5}$$

$$Z^{(j)} \succ 0, \ (j = 1, \ldots, q) \tag{6}$$

Let

$$z^{(j)} = vec\ Z^{(j)}$$

$$y^{(j)} = vec\ Y^{(j)}$$

$$B^{(j)} = \begin{bmatrix} (vec\ A_1^{(j)})^T \\ \vdots \\ (vec\ A_n^{(j)})^T \end{bmatrix}$$

$$R_p^{(j)} = Y^{(j)} - A_0^{(j)} - mat\ (B^{(j)})^T x$$

$$R_d = \begin{bmatrix} -\sum_{j=1}^{q} A_1^{(j)} \bullet Z^{(j)} \\ \vdots \\ -\sum_{j=1}^{q} A_n^{(j)} \bullet Z^{(j)} \end{bmatrix}$$

$$R_c^{(j)} = \omega_j I_{m_j} - Z^{(j)} Y^{(j)}$$

$$r_p^{(j)} = vec\ R_p^{(j)} = y^{(j)} - vec\ A_0^{(j)} - (B_j^{(j)})^T x$$

$$r_d = vec\ R_d = -\sum_{j=1}^{q} B^{(j)} z^{(j)}$$

$$r_c^{(j)} = vec\ R_c^{(j)} = vec(\omega_j I_{m_j}) - (I_{m_j} \otimes Z^{(j)}) y^{(j)}$$

$$r_c^{(j)} = vec\ R_c^{(j)} = vec(\omega_j I_{m_j}) - (Y^{(j)} \otimes I_{m_j}) z^{(j)}$$

$$G(x, y^{(1)}, \ldots, y^{(q)}, z^{(1)}, \ldots, z^{(q)}) = \begin{bmatrix} -r_p^{(1)} \\ \vdots \\ -r_p^{(q)} \\ -r_d \\ -r_c^{(1)} \\ \vdots \\ -r_c^{(q)} \end{bmatrix},$$

where *vec* is the map that stacks the columns of a matrix on top of each other into a single vector and *mat* is the inverse map. Also, let

$$r_p = \begin{bmatrix} r_p^{(1)} \\ \vdots \\ r_p^{(q)} \end{bmatrix}, r_c = \begin{bmatrix} r_c^{(1)} \\ \vdots \\ r_c^{(q)} \end{bmatrix}$$

$$y = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(q)} \end{bmatrix}, z = \begin{bmatrix} z^{(1)} \\ \vdots \\ z^{(q)} \end{bmatrix}$$

$$A = [B^{(1)}, \ldots, B^{(q)}]$$

$$E = \mathrm{diag}(Y^{(1)} \otimes I_{m_1}, \ldots, Y^{(q)} \otimes I_{m_q})$$

$$F = \mathrm{diag}(I_{m_1} \otimes Z^{(1)}, \ldots, I_{m_q} \otimes Z^{(q)})$$

$$I = \mathrm{diag}(I_{m_1^2}, \ldots, I_{m_q^2})$$

Then

$$G(x, y, z) = \begin{bmatrix} -r_p \\ -r_d \\ -r_c \end{bmatrix} = -r$$

where $r = \begin{bmatrix} r_p \\ r_d \\ r_c \end{bmatrix}$ is the residual vector.

The weighted analytic center $x_{ac}(\omega)$ for the system (1) is given as x from the root (x,y,z) of the equation

$$G(x,y,z) = 0$$

The Newton's directions $(\Delta x,\Delta y,\Delta z)$ for the equation are found by solving the system:

$$M\Delta x = AE^{-1}(Fr_p + r_c) - r_d \tag{7}$$

$$\Delta z = E^{-1}(F(r_p - A^T\Delta x) + r_c) \tag{8}$$

$$\Delta y = F^{-1}(r_c - E\Delta z) \tag{9}$$

where

$$\mathrm{M} = \mathrm{AE}^{-1}\mathrm{FA}^{\mathrm{T}}$$

An Iteration of Infeasible Newton's Method for Computing Weighted Analytic Center.

The following is an algorithm for an iteration of Infeasible Newton's Method for Weighted Analytic Center for the LMI system (1).

Step 1: Compute Newton's direction (Δx, Δy, Δz) using equations (7)-(9). This gives (Δx, Δy(1),..., Δy(q), Δz(1),..., Δz(q)).

Step 2: For each j, determine:

$$\Delta Y^{(j)} = mat\ \Delta y^{(j)}$$

$$\Delta Z^{(j)} = mat\ \Delta z^{(j)}$$

Step 3: Symmetrize ΔZ(j).

$$\text{Replace } \Delta Z^{(j)} \text{ by } \frac{1}{2}(\Delta Z^{(j)} + (\Delta Z^{(j)})^T),\ (j = 1,\ldots,q)$$

Step 4: Do line search to get stepsize h.

Step 5: Update the iterates.

$$x \leftarrow x + h\Delta x$$

$$Y^{(j)} \leftarrow Y^{(j)} + h\Delta Y^{(j)},\ (j = 1,\ldots,q)$$

$$Z^{(j)} \leftarrow Z^{(j)} + h\Delta Z^{(j)},\ (j = 1,\ldots,q)$$

Step 6: Find (x,y,z) and calculate the residual vector $r(x,y,z)$.

Any point $x \in \mathrm{IR}^n$ can be picked as a starting point. Then, for $j = 1,\ldots,q$, choose.

$$Y^{(j)} = \begin{cases} A^{(j)}(x) & A^{(j)}(x) \succ 0 \\ I_{m_j} & \text{otherwise} \end{cases}$$

$$Z^{(j)} = \omega_j (Y^{(j)})^{-1}$$

The above iteration is repeated until $\| r(x,y,z) \| < TOL$, where $r(x,y,z)$ is the residual and TOL is a given tolerance. One can use backtracking line search [10] or other techniques to get the stepsize h.

As a result of equation (4), the Infeasible Newton's method given is said to use the ZY search direction. In the next section, three other search directions for the method are described.

3. Search Directions for Infeasible Newton's Method

In this section, three other search directions for Infeasible Newton's method, namely ZY+YZ, NT and Z^{-1} directions are presented. The ZY direction has already been given in the previous section. All these methods were introduced and used in semidefinite programming [2, 12].

In the ZY method, the matrix $Z^{(j)}$ in the solution of the system (2)-(6) is not symmetric. In the ZY+YZ method, equation (4) is replaced with $Z^{(j)}Y^{(j)} + Y^{(j)}Z^{(j)} = 2\omega_j I_{m_j}$. Then $R_c^{(j)}$, $r_c^{(j)}$, E and F changes to:

$$R_c^{(j)} = \omega_j I_{m_j} - 1/2(Z^{(j)}Y^{(j)} + Y^{(j)}Z^{(j)})$$

$$r_c^{(j)} = vec\ R_c^{(j)} = vec(\omega_j I_{m_j}) - 1/2(I_{m_j} \otimes Z^{(j)} + Z^{(j)} \otimes I_{m_j})y$$

$$r_c^{(j)} = vec\ R_c^{(j)} = vec(\omega_j I_{m_j}) - 1/2(Y^{(j)} \otimes I_{m_j} + I_{m_j} \otimes Y^{(j)})z$$

$$E = \mathrm{diag}(1/2(Y^{(1)} \otimes I_{m_1} + I_{m_1} \otimes Y^{(1)}),\ldots,1/2(Y^{(q)} \otimes I_{m_q} + I_{m_q} \otimes Y^{(q)}))$$

$$F = \mathrm{diag}(1/2(I_{m_1} \otimes Z^{(1)} + Z^{(1)} \otimes I_{m_1}),\ldots,1/2(I_{m_q} \otimes Z^{(q)} + Z^{(q)} \otimes I_{m_q}))$$

In the Z^{-1} method, equation (4) is replaced with $Y^{(j)} = \omega_j (Z^{(j)})^{-1}$. Then $R_c^{(j)}$, $r_c^{(j)}$, E and F become:

$$R_c^{(j)} = \omega_j (Z^{(j)})^{-1} - Y^{(j)}$$

$$r_c^{(j)} = vec\ R_c^{(j)} = \omega_j vec((Z^{(j)})^{-1}) - vec(Y^{(j)})$$

$$= \omega_j vec((Z^{(j)})^{-1}) - (I_{m_j} \otimes I_{m_j})vec(Y^{(j)})$$

$$E = \mathrm{diag}(\omega_1((Z^{(1)})^{-1} \otimes (Z^{(1)})^{-1}),\ldots,\omega_q((Z^{(q)})^{-1} \otimes (Z^{(q)})^{-1}))$$

$$F = \mathrm{diag}(I_{m_1} \otimes I_{m_1},\ldots,I_{m_q} \otimes I_{m_q})$$

The NT method replaces E and F in the Z^{-1} method with

$$E = \mathrm{diag}(\omega_1((W^{(1)})^{-1} \otimes (W^{(1)})^{-1}),\ldots,\omega_q((W^{(q)})^{-1} \otimes (W^{(q)})^{-1}))$$

$$F = \mathrm{diag}(I_{m_1} \otimes I_{m_1},\ldots,I_{m_q} \otimes I_{m_q}),$$

where for each j, $W^{(j)}$ is the unique scaling matrix defined by

$$W^{(j)} = (Z^{(j)})^{\frac{1}{2}}((Z^{(j)})^{\frac{1}{2}}Y^{(j)}(Z^{(j)})^{\frac{1}{2}})^{-\frac{1}{2}}(Z^{(j)})^{\frac{1}{2}}$$

In the case of ZY+YZ, Z^{-1} and NT methods, both $Z^{(j)}$ and $Y^{(j)}$ are symmetric [2]. See Theorem 3.1.

Theorem 3.1 *In the ZY+YZ, Z^{-1} and NT methods, both $Z^{(j)}$ and $Y^{(j)}$ are symmetric.*

Proof: Since each $A_i^{(j)}$ is symmetric, then by equation (2), it is clear that each $Y^{(j)}$ is symmetric. In the $ZY + YZ$ method, equation (4), that is, $Z^{(j)}Y^{(j)} + Y^{(j)}Z^{(j)} = 2\omega_j I_{mj}$ is a symmetric matrix equation. So, equations (2)-(6) and $G(x,y,z)=0$ result in a mapping G for Newton's method with the same domain and range. In the Z^{-1} method, equation (4), that is, $Y^{(j)} = \omega_j(Z^{(j)})^{-1}$ is a symmetric matrix equation. So, equations (2)-(6) and $G(x,y,z)=0$ result in a mapping G for Newton's method having the same domain and range. Hence, application of Newton's method in ZY+YZ and Z^{-1} methods leads to symmetric interates $Z^{(j)}$. The proof of symmetry of $Z^{(j)}$ in the NT method follows from Theorem 3.1 in a paper given by Todd et al. [12].

4. Numerical Experiments

In this section, numerical experiments are done to compare ZY, ZY+YZ, Z^{-1} and NT search methods in the Infeasible Newton's method for computing weighted analytic center.

Table 1 describes the 35 random test problems used for our numerical experiments. These are the same test problems used by Jibrin in a 2015 paper [4], with the exception of Problem 35. The second column of Table 1 gives the dimension n of the ambient space and the third column is the number q of LMI constraints. The dimensions m_j of the matrices are given in the fourth column. For each problem, n, q and m_j are random integers in the intervals [2, 30], [1, 10] and [1, 5] respectively. For each j, the LMI $A_0^{(j)} + \sum_{i=1}^{n} x_i A_i^{(j)} \succeq 0$ was generated randomly as follows: $A_0^{(j)}$ is an $m_j \times m_j$ diagonal matrix with each diagonal entry chosen from $U(0,1)$. Each $A_i^{(j)}$ ($1 \le i \le n$) is a random $m_j \times m_j$ symmetric and sparse matrix with approximately $0.8 * m_j^2$ nonzero entries generated using the Matlab command $sprandsym(m_j, 0.8)$. Each problem has a nonempty interior.

All codes used in our experiments were written in Matlab and ran on Dell OPTIPLEX 880 computer. Infeasible Newton's method codes for weighted analytic center was ran using the four search directions: ZY, ZY+YZ, Z^{-1} and NT. All four methods were implemented using

Table 1. *Test Problems.*

Test Problem	*n*	*q*	*m*
1	2	2	[2,1]
2	3	4	[3,4,1,2]
3	2	2	[2,2]
4	5	3	[4,1,3]
5	4	3	[5,4,3]
6	4	5	[4,3,1,1,4]
7	3	3	[4,2,3]
8	3	4	[4,2,2,5]
9	5	3	[4,1,1]
10	3	5	[5,3,5,1,4]
11	2	7	[2,5,3,5,2,5,1]
12	5	6	[5,1,3,4,1,4]]
13	14	5	[5,1,3,4,2]
14	20	5	[5,2,5,1,5]
15	3	8	[5,4,1,5,3,5,1,3]
16	9	7	[1,4,2,4,4,2,2]
17	6	5	[4,4,2,1,4]
18	10	2	[3,5]
19	15	9	[2,5,3,1,2,3,3,1,2]
20	8	2	[4,5]
21	19	7	[5,2,2,2,5,5,5]
22	9	10	[3,4,1,1,3,5,5,4,5,2]
23	3	4	[2,3,2,5]
24	8	2	[5,1]
25	2	8	[5,2,1,1,1,5,3,3]
26	13	8	[4,1,4,2,3,1,2,1]
27	24	10	[5,4,5,1,4,2,3,5,5,2]
28	5	6	[4,1,4,2,1,3]
29	16	3	[2,2,3]
30	2	2	[4,5]
31	2	4	[1,5,5,5]
32	4	4	[5,1,4,5]
33	4	4	[1,2,3,5]
34	17	9	[1,5,2,1,2,5,1,4,3]
35	4	6	[2,1,5,3,5,2]

Table 2. *Iterations and time taken by each method to find a point in the interior of the feasible region using the given weights. The entry "*" means that Infeasible Newton's method has failed to find an interior point after the maximum number of 500 iterations or due to numerical problems.*

Prob	Weights	ZY		Z^{-1}		ZY+YZ		NT	
		Iter	Time	Iter	Time	Iter	Time	Iter	Time
	Ω		(sec)		(sec)		(sec)		(sec)
1	[10^{12},10]	2	0.0490	*	*	2	0.2311	*	*
2	[10^{12},100,100,1]	3	0.0271	*	*	3	0.0280	*	*
3	[10^{12},1000]	51	0.2100	*	*	48	0.2970	*	*
4	[10^{12},10,1]	20	0.1644	*	*	17	0.1550	18	0.4929
5	[10^{12},1,10]	2	0.0167	*	*	2	0.0130	*	*
6	[1,10^{12},1,10,100]	*	*	*	*	*	*	2	0.1377
7	[100,10,10^{12}]	129	2.2922	*	*	*	*	*	*

Prob	Weights	ZY		Z^{-1}		ZY+YZ		NT	
		Iter	Time	Iter	Time	Iter	Time	Iter	Time
8	[1,1000,10^{12},10]	*	*	*	*	*	*	*	*
9	[10^{12},1000,1000]	1	0.0089	1	0.0040	1	0.0071	*	*
10	[10^{12},1000,100,1000,100]	6	0.1059	*	*	6	0.0612	52	2.8783
11	[10^{12},10,100,1000,1000,10,100]	*	*	*	*	*	*	*	*
12	[10^{12},1,100,10,1,10]	3	0.0480	*	*	3	0.0338	*	*
13	[1000,10^{12},1000,100,100]	*	*	*	*	*	*	*	*
14	[1,1000,10^{12},1000,1]	*	*	*	*	*	*	*	*
15	[1,1,1,100,10^{12},100,100,10]	*	*	*	*	*	*	*	*
16	[10,1,10,1000,1000,10,1]	24	0.3132	*	*	23	0.2861	*	*
17	[100,100,10,1,1000]	3	0.0316	*	*	3	0.0275	*	*
18	[1,1000]	2	0.2249	*	*	2	0.0098	130	0.5293
19	[100,10,1000,1,1000,100,100,1000,10]	10	0.1678	*	*	10	0.1590	*	*
20	[1000,1000]	1	0.0084	1	0.0086	1	0.0061	*	*
21	[1,1,10,1,100,10,1]	21	0.3363	*	*	21	0.3405	*	*
22	[1,1000,100,10,1000,1,100,100,100,1]	7	0.1721	*	*	4	0.0917	*	*
23	[100,10,1,1]	4	0.0344	*	*	4	0.0292	5	0.0614
24	[1000,1]	12	0.0536	*	*	11	0.0423	*	*
25	[1,1,10,1,1000,1,1,10]	8	0.1268	*	*	9	0.1312	*	*
26	[100,1,100,1,1,1,1,1]	20	0.2687	*	*	20	0.2713	*	*
27	[1,1,1,10,1,10,1,100,1,10]	8	0.2345	*	*	10	0.2790	*	*
28	[10,10,100,1,1,1]	7	0.0773	*	*	7	0.0672	*	*
29	[1,1,10]	1	0.0122	1	0.0043	1	0.0077	1	0.0061
30	[10,1]	2	0.0147	*	*	2	0.0098	*	*
31	[1,10,1,1]	1	0.0136	1	0.0120	1	0.0098	*	*
32	[100,1,10,1]	8	0.0759	*	*	8	0.0647	11	0.1767
33	[100,100,1,10]	6	0.0495	*	*	6	0.0414	2	0.0116
34	[1,100,1,1,1,10,100,100,10]	7	0.1372	*	*	8	0.1404	*	*
35	[1,1,100,1,10,1]	2	0.0277	*	*	2	0.0246	*	*

Table 3. *Iterations and time taken by each method to find the weighted analytic center using the given weights. The entry " ★ " means that Infeasible Newton's method has failed to find the weighted analytic center due to numerical problems or after the maximum number of 500 iterations.*

Prob	Weights	ZY		Z^{-1}		ZY+YZ		NT	
		Iter	Time	Iter	Time	Iter	Time	Iter	Time
	Ω		(sec)		(sec)		(sec)		(sec)
1	[10^{12},10]	6	0.0590	*	*	8	0.2883	*	*
2	[10^{12},100,100,1]	*	*	*	*	*	*	*	*
3	[[10^{12},1000]	97	0.3827	*	*	93	0.5589	*	*
4	[10^{12},10,1]	*	*	*	*	*	*	*	*
5	[10^{12},1,10]	*	*	*	*	*	*	*	*
6	[1,10^{12},1,10,100]	*	*	*	*	*	*	*	*
7	[100,10,10^{12}]	*	*	*	*	*	*	*	*
8	[1,1000,10^{12},10]	*	*	*	*	*	*	*	*
9	[10^{12},1000,1000]	*	*	*	*	*	*	*	*
10	[10^{12},1000,100,1000,100]	*	*	*	*	*	*	*	*
11	[10^{12},10,100,1000,1000,10,100]	*	*	*	*	*	*	*	*
12	[10^{12},1,100,10,1,10]	*	*	*	*	*	*	*	*
13	[1000,10^{12},1000,100,100]	*	*	*	*	*	*	*	*
14	[1,1000,10^{12},1000,1]	*	*	*	*	*	*	*	*
15	[1,1,1,100,10^{12},100,100,10]	*	*	*	*	*	*	*	*
16	[10,1,10,1000,1000,10,1]	27	0.3412	*	*	26	0.3146	*	*
17	[100,100,10,1,1000]	7	0.0612	*	*	7	0.0568	*	*
18	[1,1000]]	8	0.2463	*	*	7	0.0273	*	*
19	[100,10,1000,1,1000,100,100,1000,10]	17	0.2600	*	*	18	0.2690	*	*
20	[1000,1000]	5	0.0254	*	*	5	0.0206	*	*
21	[1,1,10,1,100,10,1]	24	0.3748	*	*	23	0.3687	*	*
22	[1,1000,100,10,1000,1,100,100,100,1]	10	0.2307	*	*	10	0.2137	*	*
23	[100,10,1,1]	9	0.0619	*	*	8	0.0511	*	*
24	[1000,1]	14	0.0602	*	*	13	0.0487	*	*
25	[1,1,10,1,1000,1,1,10]	11	0.1606	*	*	11	0.1542	*	*
26	[100,1,100,1,1,1,1,1]	22	0.2893	*	*	22	0.2922	*	*
27	[1,1,1,10,1,10,1,100,1,10]	12	0.3293	*	*	13	0.3511	*	*
28	[10,10,100,1,1,1]	10	0.1016	*	*	9	0.0836	*	*
29	[1,1,10]	*	*	*	*	*	*	*	*
30	[10,1]	6	0.0304	*	*	6	0.0246	*	*
31	[1,10,1,1]	5	0.0458	149	1.5269	4	0.0335	*	*

Prob	Weights	ZY		Z^{-1}		ZY+YZ		NT	
		Iter	Time	Iter	Time	Iter	Time	Iter	Time
32	[100,1,10,1]	12	0.1051	*	*	11	0.0851	*	*
33	[100,100,1,10]	9	0.0668	*	*	10	0.0639	*	*
34	[1,100,1,1,10,100,100,10]	10	0.1726	*	*	11	0.1817	*	*
35	[1,1,100,1,10,1]	6	0.0390	*	*	6	0.0607	*	*

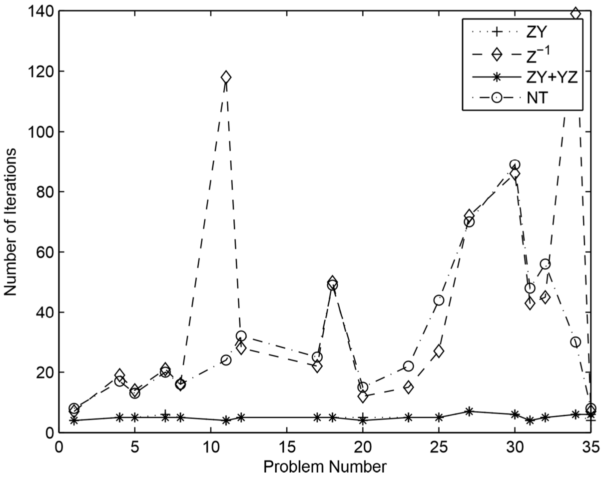

Figure 1. *Problem Number Vs Iterations needed to find analytic center for the 18 problems where all four methods were successful. +=ZY,* $\diamond = Z^{-1}$*, *=ZY+YZ, o=NT.*

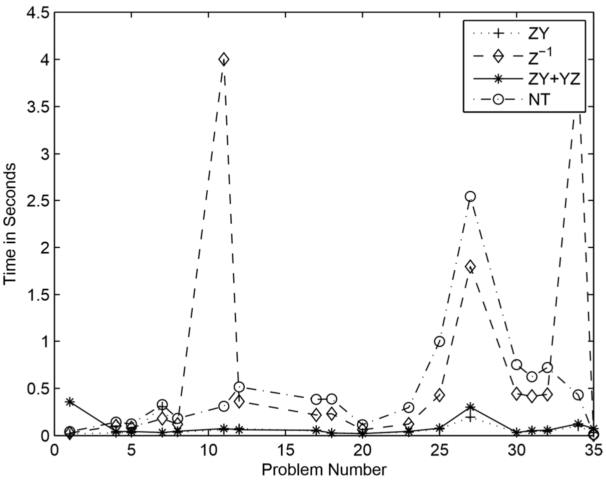

Figure 2. *Problem Number Vs Time needed to find analytic center for the 18 problems where all four methods were successful. +=ZY,* $\diamond = Z^{-1}$ *, *=ZY+YZ, o=NT.*

Table 4. *Number of problems out of 35, where the given method was successful from the results in Table 2 and Table 3.*

Method	Finding Interior Point	Finding Weighted Analytic Center
	Success Number	Success Number
ZY	29	21
Z^{-1}	4	1
ZY+YZ	28	21
NT	8	0

a tolerance of $TOL = 10^{-4}$ and up to a maximum of 500 iterations. The starting point is infeasible and random such that each of its components is chosen from a normal distribution with mean 0 and variance 10^6. Backtracking line search technique is used in each of the four methods. Table 2 and Table 3 give the results of our experiments, where one of the weights is relatively very large in Problems 1-15. In Problems 16-35, one of the weights is relatively large, but not very large. Table 2 gives the number of iterations and time to find an interior point of the feasible region from the starting point. In Table 2, the entry "$\star$" means that Infeasible Newton's method has failed to find an interior point after the maximum number of 500 iterations or due to numerical problems. We see that ZY+YZ takes the least number of iterations on average and ZY takes the least time on average. ZY+YZ and ZY are the best methods, followed by NT and then Z^{-1}. Table 3 gives the number of iterations and time to find the weighted analytic center. In Table 3, the entry "$\star$" means that Infeasible Newton's method has failed to find the weighted analytic center due to numerical problems or after the maximum number of 500 iterations. Table 4 summarizes the results of Table 2 and Table 3. The results in Table 4 show that ZY+YZ and ZY were the most successful while both Z^{-1} and NT failed to work well. In Table 5 and Table 6, the results are given where each of the weights is 1. In this case, the weighted analytic center is simply called the analytic center.

Table 5. *Iterations and time taken by each method to find a point in the interior of the feasible region using the given the weight* $\omega = [1,1,...,1]$*. The entry "*$\star$*" means that Infeasible Newton's method has failed to find an interior point after the maximum number of 500 iterations or due to numerical problems.*

Problem	ZY		Z^{-1}		ZY+YZ		NT	
	Iter	Time	Iter	Time	Iter	Time	Iter	Time
		(sec)		(sec)		(sec)		(sec)
1	1	0.0089	1	0.0073	1	0.3195	1	0.0079
2	2	0.0177	2	0.0188	2	0.0213	2	0.0168
3	1	0.0070	1	0.0055	1	0.0088	1	0.0078
4	2	0.0166	2	0.0151	2	0.0218	2	0.0191
5	1	0.0103	1	0.0097	1	0.0119	1	0.0116
6	1	0.0133	1	0.0134	1	0.0156	1	0.0090
7	1	0.2494	1	0.0762	1	0.0091	1	0.1367
8	1	0.0117	1	0.0136	1	0.0126	1	0.0253
9	1	0.2908	1	0.0773	1	0.0095	1	0.0489
10	1	0.0070	1	0.0068	1	0.0164	1	0.0159
11	1	0.0201	1	0.0169	1	0.0219	1	0.0115

Problem	ZY		Z^{1}		ZY+YZ		NT	
	Iter	Time	Iter	Time	Iter	Time	Iter	Time
12	2	0.0458	2	0.0613	2	0.0295	2	0.0469
13	2	0.0293	3	0.0435	2	0.0359	2	0.0146
14	13	0.1498	*	*	13	0.1876	24	0.1567
15	1	0.0237	1	0.0235	1	0.0239	1	0.0222
16	2	0.0446	2	0.0375	2	0.0423	2	0.0181
17	2	0.0256	2	0.0246	2	0.0273	2	0.0325
18	2	0.0151	2	0.0144	2	0.0132	2	0.0184
19	3	0.0559	2	0.0448	3	0.0579	*	*
20	1	0.0087	1	0.0084	1	0.0084	1	0.0092
21	12	0.2475	10	0.1670	12	0.2669	*	*
22	2	0.0553	*	*	2	0.0605	2	0.0703
23	1	0.0111	1	0.0121	1	0.0124	1	0.0149
24	4	0.0229	*	*	5	0.0349	36	0.3229
25	2	0.0346	2	0.0394	2	0.0363	2	0.0462
26	13	0.4362	*	*	13	0.2624	22	0.5945
27	3	0.1036	2	0.0788	3	0.1366	3	0.1310
28	3	0.0375	3	0.0388	3	0.0419	3	0.0450
29	1	0.0116	1	0.0039	1	0.0115	1	0.0053
30	2	0.0145	2	0.0140	2	0.0131	2	0.0201
31	1	0.0223	1	0.0130	1	0.0159	1	0.0157
32	2	0.0222	2	0.0233	2	0.0257	2	0.0286
33	4	0.0349	*	*	4	0.0461	24	0.2680
34	2	0.0453	2	0.0464	2	0.0524	2	0.0276
35	1	0.0270	1	0.0026	2	0.0300	1	0.0018

Table 6. *Iterations and time taken by each method to find the analytic center that is, using the given the weight $\omega = [1,1,...,1]$. The entry "★" means that Infeasible Newton's method has failed to find the analytic center due to numerical problems or after the maximum number of 500 iterations.*

Problem	ZY		Z^{1}		ZY+YZ		NT	
	Iter	Time (sec)	Iter	Time (sec)	Iter	Time (sec)	Iter	Time (sec)
1	4	0.0181	7	0.0296	4	0.3564	8	0.0433
2	6	0.0404	*	*	6	0.0534	47	0.4401
3	10	0.0381	*	*	9	0.0509	19	0.3052
4	5	0.0299	19	0.1036	5	0.0424	17	0.1428
5	5	0.0307	14	0.0831	5	0.0428	13	0.1241
6	5	0.0435	*	*	5	0.0575	29	0.2544
7	6	0.3064	21	0.1810	5	0.0331	20	0.3278
8	5	0.0352	16	0.1205	5	0.0465	16	0.1821
9	5	0.3294	*	*	5	0.0312	62	0.4482
10	7	0.0399	85	0.5903	5	0.0635	70	0.5882
11	4	0.0618	118	4.0018	4	0.0734	24	0.3111
12	5	0.0743	28	0.3619	5	0.0626	32	0.5154
13	6	0.0638	*	*	6	0.0850	50	0.4806
14	17	0.1873	*	*	16	0.2343	101	0.7694
15	6	0.1046	*	*	5	0.0999	87	1.8125
16	6	0.0886	*	*	6	0.0919	32	0.4146
17	5	0.0484	22	0.2176	5	0.0564	25	0.3839
18	5	0.0277	50	0.2313	5	0.0266	49	0.3879
19	7	0.1060	*	*	7	0.1317	*	*
20	5	0.0258	12	0.0627	4	0.0219	15	0.1141
21	16	0.3230	*	*	16	0.3482	*	*
22	6	0.1344	*	*	6	0.1682	79	3.5432
23	5	0.0355	15	0.1218	5	0.0439	22	0.2976
24	8	0.0377	*	*	8	0.0503	*	*
25	5	0.0707	27	0.4286	5	0.0777	44	0.9968
26	16	0.4689	*	*	16	0.3070	44	1.0047
27	7	0.1946	72	1.7975	7	0.3018	70	2.5422
28	6	0.0624	*	*	6	0.0733	55	0.2527
29	*	*	*	*	*	*	*	*
30	6	0.0325	86	0.4434	6	0.0323	89	0.7516
31	4	0.0460	43	0.4163	4	0.0535	48	0.6225
32	5	0.0449	45	0.4357	5	0.0559	56	0.7213
33	8	0.0593	*	*	8	0.0837	*	*
34	6	0.0998	139	3.8625	6	0.1242	30	0.4312
35	4	0.0311	7	0.0079	6	0.0740	8	0.0128

Table 7. *Number of problems out of 35, where method was successful from the results in Table 5 and Table 6.*

Method	Finding Interior Point	Finding Analytic Center
	Success Number	**Success Number**
ZY	35	34
Z^{-1}	30	19
ZY+YZ	35	34
NT	33	30

Table 5 gives the number of iterations and time to find an interior point of the feasible region with each weight set as 1. It shows that ZY+YZ and ZY are the best methods in finding an interior in terms of iterations and time, followed by NT, and then by Z^{-1}. Table 6 gives the number of iterations and time to find the analytic center. Table 7 summarizes the results of Table 5 and Table 6. Note all the four methods were successful in finding the analytic center in 18 of the 35 test problems. Figure 1 shows the graph of iterations vs. the 18 problems. The graph of time vs. the 18 problems is given in Figure 2. Figure 1, Figure 2 and the results in Table 7 show that ZY+YZ and ZY are the best methods, followed by NT, and then Z^{-1}.

5. Conclusion

Four search direction methods for Infeasible Newton's method for computing weighted analytic center for linear matrix inequalities, namely: ZY, ZY+YZ, Z^{-1} and NT methods are presented and compared. The four search directions ZY, ZY+YZ, Z^{-1} and NT methods have been used in the problem of semidefinite programming.

Randomly generated test problems are used to compare the four methods. Our numerical results indicate that the ZY+YZ and ZY methods converge more rapidly and they handle weights better compared to the other methods, when some of the weights are very large relative to the other weights. ZY+YZ took the least number of iterations on average and is closely followed by ZY, then NT and then Z^{-1} methods. ZY also took the least time among the other four methods.

Backtracking line search is used in our experiments. It would be of interest to study and compare the effect of other line search methods on the four methods. We hope in future to compare the performance of the four methods on weighted analytic center for second order cone constraints.

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