Applied and Computational Mathematics
2016; 5(6): 230-233
http://www.sciencepublishinggroup.com/j/acm
doi: 10.11648/j.acm.20160506.11
ISSN: 2328-5605 (Print); ISSN: 2328-5613 (Online)

The Average Dwell Time Condition $\tau_a > \dfrac{\ln\mu}{\alpha}$ Is Necessary & Sufficient for Arbitrary Switching Stability of Switched Nonlinear Systems

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To cite this article:
Jiqiang Wang. The Average Dwell Time Condition $\tau_a > \dfrac{\ln\mu}{\alpha}$ Is Necessary & Sufficient for Arbitrary Switching Stability of Switched Nonlinear Systems. *Applied and Computational Mathematics*. Vol. 5, No. 6, 2016, pp. 230-233. doi: 10.11648/j.acm.20160506.11

Received: October 12, 2016; **Accepted**: November 14, 2016; **Published**: December 8, 2016

Abstract: Constrained switching of switched nonlinear systems consists of many classes of switching signals with markedly different features. One of the most important ones is the average dwell time (ADT) switching. For switched systems, it is a well-known result that a switched nonlinear system is globally uniformly asymptotically stable under *arbitrary switching sequence* if the ADT satisfies the lower bound defined by a real constant value ($\ln\mu/\alpha$). In this note, it will be shown that this ADT condition is also necessary.

Keywords: Switched Nonlinear Systems, Multiple Lyapunov Functions, Average Dwell Time, Arbitrary Switching Stability

1. Introduction

The theory of switched systems has matured into an important field of research. Its development has been motivated by the fact that many dynamical systems can be represented by a series of subsystems or modes, scheduled by switches among these modes. Switched systems can be classified into stated-controlled switched systems and time-controlled switched systems. The former refers to those systems where switch is triggered by system states satisfying certain prescribed conditions (guard conditions), while the latter implies that the system is switched according to a time sequence. For both classes of switched systems, abounding examples exist in biological and engineering systems.

For dynamical systems, stability is an important issue for investigation of the time-evolutionary properties. One of the important problems concerns the theory of stability under arbitrary switching, namely, the switched system retains stability for any switching signal. This arbitrary switching stability is important due to the fact that many engineering systems are required to possess such a property. Indeed, for systems experiencing frequent switches, arbitrary switching stability is desirable or even compulsory. For example, for aircraft engines working over a large flight envelope, their control systems need to be switched among a variety of controllers. It is thus desirable to have an arbitrary switching stability property, although the controllers often experience switches over a (prescribed) sequence of acceleration and deceleration schedules [1]. In fact, some unfortunate switching still poses threats to deterioration of aircraft engine performance [2]. For electric power grid, however, stability should be retained under any type of switching signals as well as for any switching sequence of signals. Thus arbitrary switching stability must be kept as an a priory requirement for safe and reliable operation of power systems.

Consequently, arbitrary switching stability is one of the important topics in the field of switched nonlinear systems. Many fundamentally important results have been obtained during the past decades [3-7]. To guarantee stability under arbitrary switching, the common Lyapunov function method plays an important role. This is because the existence of a common Lyapunov function implies the global uniform asymptotic stability (GUAS) of the switched system for any switching sequence. Indeed, the importance of the existence of a common Lyapunov function is actually consolidated by a converse theorem, dictating that if a switched system is GUAS, then all the subsystems share a common Lyapunov function [8]. However it is also generally recognized that the common Lyapunov function approach to guarantee GUAS under arbitrary switching is very conservative and in many applications the average dwell time (ADT) switching is meaningful and flexible. The concept of ADT switching is introduced in [9] and means that the number of switches in a finite time interval is bounded and the average time between consecutive switching is not less than a constant. Thus ADT switching precludes randomly fast switching signals and avoids Zeno behaviour.

2. Preliminary and Main Result

For ADT switching of switched nonlinear systems, it is well known that a sufficient condition exists to guarantee the system is GUAS for any switching signal with ADT. In specific, it is stated as follows [9]:

Theorem 1: Consider the switched system $\dot{x}_t = f_\sigma(x_t)$ *, and let $\alpha > 0$, $\mu > 1$ be given constants. Suppose that there exist C^1 functions $V_{\sigma(t)} : \Re^N \to \Re$, $\sigma(t) \in \ell$, and two K_∞ functions k_1 and k_2 such that $\forall \sigma(t) = i$, $k_1(|x_t|) \le V_i(x_t) \le k_2(|x_t|)$, $\dot{V}_i(x_t) \le -\alpha V_i(x_t)$, and furthermore $\forall (i,j) \in \ell \times \ell$, $i \ne j$, $V_i(x_t) \le \mu V_j(x_t)$; then the system is GUAS for any switching signal with average dwell time $\tau_a > \tau_a^* = \dfrac{\ln\mu}{\alpha}$.

Remark 1: This result shows that the ADT condition is sufficient for arbitrary switching stability of switched (linear and nonlinear) systems. In this note it will be shown that the ADT condition $\tau_a > \tau_a^* = \dfrac{\ln\mu}{\alpha}$ is also necessary for switched systems to be of arbitrary switching stability. To the best of the author's knowledge, the necessity of the ADT condition has not been recognized. Showing the ADT condition is "if and only if" is thus an important progress in the stability theory of switched nonlinear systems.

Given the above preliminary, the main result can now be stated below:

Theorem 2: Consider the switched system $\dot{x}_t = f_\sigma(x_t)$ and let $\alpha > 0$, $\mu > 1$ be given constants. Suppose that there exist C^1 functions $V_{\sigma(t)} : \Re^N \to \Re$, $\sigma(t) \in \ell$, and two K_∞ functions k_1 and k_2 such that $\forall \sigma(t) = i$, $k_1(|x_t|) \le V_i(x_t) \le k_2(|x_t|)$, $\dot{V}_i(x_t) \le -\alpha V_i(x_t)$, and $\forall (i,j) \in \ell \times \ell$, $i \ne j$, $V_i(x_t) \le \mu V_j(x_t)$; then the system is GUAS for any switching signal *if and only if t*he ADT satisfies the condition $\tau_a > \tau_a^* = \dfrac{\ln\mu}{\alpha}$.

Proof: The sufficiency part is well-known and we show the necessity part below. We do this by first considering a generalized Lyapunov-like function, allowing the energy function to increase to a limited extent. From this general situation, we derive a "fast" switching rule in contrast to the usual "slow" switching rule as stated in Theorem 1. We then show that once the generalized Lyapunov-like functions become the C^1 functions $V_{\sigma(t)} : \Re^N \to \Re$ defined above, then the *minimum dwell time* among all the switching sequences is exactly the $\tau_a^* = \dfrac{\ln\mu}{\alpha}$. This implies that the ADT condition $\tau_a > \tau_a^*$ is in fact tight. That is, to guarantee the system to be GUAS for any switching signal, the ADT has to satisfy $\tau_a > \tau_a^*$, hence the necessity of the ADT condition is proved. The line of thought is delineated below.

Step 1: Direct proof can be very difficult. Here an approach motivated by [10] and further exposed in [11] is adopted where a so-called week Lyapunov function is defined. This allows the Lyapunov-like function to rise to a limited extent and thus is very general. Now consider $\sigma(t) = i$ and within the interval $[t_i, t_{i+1})$, denote the unions of scattered subintervals during which the week Lyapunov function is increasing and decreasing by $T_r(t_i, t_{i+1})$ and $T_d(t_i, t_{i+1})$, respectively. Hence $[t_i, t_{i+1}) = T_r(t_i, t_{i+1}) \cup T_d(t_i, t_{i+1})$. Further use $T_r(t_{i+1} - t_i)$ and $T_d(t_{i+1} - t_i)$ to represent the length of $T_r(t_i, t_{i+1})$ and $T_d(t_i, t_{i+1})$ correspondingly. Then we have the following result:

Lemma 1: Consider the switched system $\dot{x}_t = f_\sigma(x_t)$, and let $\alpha > 0$, $\beta > 0$ and $\mu > 1$ are prescribed constants. If there exist smooth functions $V_{\sigma(t)} : \Re^n \to \Re$ and two K_∞ functions k_1 and k_2 such that for each $\sigma(t) = i$, the following conditions hold:

$$k_1(|x_t|) \le V_i(x_t) \le k_2(|x_t|)$$

$$\dot{V}_i(x_t) \le \begin{cases} -\alpha V_i(x_t) & over \quad t \in T_d(t_i, t_{i+1}) \\ \beta V_i(x_t) & over \quad t \in T_r(t_i, t_{i+1}) \end{cases}$$

$$V_i(x_t) \le \mu V_j(x_t) \qquad \forall \big(\sigma(t) = i \quad \& \quad \sigma(t^-) = j\big) \in \ell \times \ell, i \ne j$$

then the system is GUAS for any switching signal with ADT

$$\tau_a < \frac{(\alpha + \beta) T_{\min} - \ln\mu}{\beta}, \ T_{\min} = \min T_d(t_i - t_{i-1}), \qquad \forall i \in \ell$$

Proof: For $t \in [t_i, t_{i+1})$, we have:

* σ is a piecewise constant function of time, called a switching signal, taking values in a finite set $\ell = \{1, \cdots N\}$. The other notation used in this note is fairly standard and will not be explicitly defined without confusion.

$$V_i(x_t) \le e^{-\alpha T_d(t_i,t)+\beta T_r(t_i,t)} V_i(x_{t^-})$$
$$\le e^{-\alpha T_d(t_i,t)-\beta T_d(t_i,t)+\beta T_d(t_i,t)+\beta T_r(t_i,t)} V_i(x_{t^-})$$
$$\le e^{\beta(t-t_i)} e^{-(\alpha+\beta)T_d(t_i,t)} V_i(x_{t^-})$$
$$\le e^{\beta(t-t_i)} e^{-(\alpha+\beta)T_{\min}} \mu V_{i-1}(x_{t^+})$$
$$\le e^{\beta(t-t_0)} \left(e^{-(\alpha+\beta)T_{\min}}\right)^{N_\sigma(t_0,t)} \mu^{N_\sigma(t_0,t)} V_0(x_{t_0})$$
$$\le e^{\{N_\sigma(t_0,t)[\ln\mu-(\alpha+\beta)T_{\min}]+\beta(t-t_0)\}} V_0(x_{t_0})$$

where we have made the definition $T_{\min} \equiv \min_l T_d(t_i - t_{i-1})$, that is the minimum decreasing interval over any switching sequence. Hence if $N_\sigma(t_0,t)\left[\ln\mu-(\alpha+\beta)T_{\min}\right]+\beta(t-t_0)<0$, then $V_i(x_t)$ will be decreasing and the system will achieve GUAS. Now the condition $N_\sigma(t_0,t)\left[\ln\mu-(\alpha+\beta)T_{\min}\right]+\beta(t-t_0)<0$ is exactly the average dwell time defined by $\tau_\alpha \equiv \dfrac{t-t_0}{N_\sigma(t_0,t)}$. This completes the proof.

The significance of Lemma 1 lies in the fact that the weak Lyapunov function is very general and incorporates the Lyapunov-like functions defined in Theorem 2. We now turn to the implication of the result in step 2.

Step 2: It is seen that lemma 1 is a "fast switching" result, in contract to the usual "slow switching" that bounds the number of switches in a finite time interval. Fast switching result is desirable because a system will achieve arbitrary switching stability if the upper bound for fast switching is larger than the lower bound for slow switching. Now consider $\beta = 0$, that is, over any interval $[t_i, t_{i+1})$ the Lyapunov-like function $V_i(x_t)$ is non-increasing, then lemma 1 tells that the only requirement for fast switching stability is the nominator $(\alpha+\beta)T_{\min} - \ln\mu \ge 0$, that is: $T_{\min} \ge \dfrac{\ln\mu}{\alpha}$. It can then deduce from the definition $T_{\min} \equiv \min T_d(t_i - t_{i-1}),\quad \forall i \in \ell$ that the minimum decreasing duration over any interval should satisfy $T_{\min} \ge \tau^* = \dfrac{\ln\mu}{\alpha}$. This is equivalently to say that the minimum dwell time over any switching sequence should be at least $\tau^* = \dfrac{\ln\mu}{\alpha}$.

Step 3: Now the case $\beta = 0$ is exactly the Lyapunov-like function defined in Theorem 2. To recap, the sufficiency part says that the system is GUAS for any switching signal *if* the ADT satisfies the condition $\tau_a > \tau_a^* = \dfrac{\ln\mu}{\alpha}$; the result from step 2 shows that to guarantee GUAS, the minimum dwell time over any switching sequence should be at least $\tau^* = \dfrac{\ln\mu}{\alpha}$. That is the estimation $\tau_a > \tau_a^* = \dfrac{\ln\mu}{\alpha}$ is actually tight, demonstrating the necessity of the ADT condition. □

Remark 2: The proof given here is not very straightforward but the approach is illuminating in that it provides insight into the nature of switching systems, e.g. in the proof of Theorem 2, we obtain $T_{\min} \ge \ln\mu/\alpha$, which implies that to compensate the "jump" $V_i(x_t) \le \mu V_j(x_t)$ ($\mu > 1$), the minimum dwell time over all switching signals should be at least $\ln\mu/\alpha$; while $\mu = 1$ reduces to $T_{\min} \ge 0$, which is obviously true as this is simply the case of usual multiple Lyapunov functions with switch matching inequality.

Remark 3: The result in lemma 1 can be reformulated in a slow switching fashion resulting in $\tau_a > \dfrac{(\alpha+\beta)T_{\max} + \ln\mu}{\alpha}$, $T_{\max} = \max_l T_r(t_i - t_{i-1})$, $\beta = 0$ implies no increasing in the Lyapunov-like function and hence $T_{\max} = 0$. The ADT condition then becomes $\tau_a > \dfrac{\ln\mu}{\alpha}$, the usual ADT condition.

Remark 4: Consequently, switched systems can actually be characterized by slow switching and fast switching. This implies that two mechanisms exist for stability of constrained switched systems. This provides a new perspective towards an important issue in natural systems, namely how a system can maintain long-term stability while experiencing short-term instability.

3. Discussions

The sufficiency of the ADT condition has been proved to be necessary as well. However, it must be warned that such a necessary and sufficient condition must be explained within the background of constrained switching. That is, the switching signals are required to possess a dwell time property. And in this sense, arbitrary switching stability does not imply that the switching signals are of any type in this note but constrained within dwell time signals. Even with this restriction, the result presented here is of significance. Meanwhile, as the procedures of the proof involves in deploying Lyapunov-like functions, it is expected that the results are applicable to other systems such as linear and nonlinear switched descriptor systems [12-15], with possible development into fractional order switched systems [16, 17] etc.

While it is claimed that the results obtained in this note are of theoretical importance, they also have significance for practical engineering systems. For example, many systems are controlled by digital controllers, thus it is desirable to implement control signals as dwell time switching ones. And once the control signals satisfy the corresponding dwell time condition dictated in Theorem 2, the closed loop system is guaranteed to be arbitrary switching stability.

4. Conclusion

Arbitrary switching stability under ADT is an important class of switching signals for switched nonlinear systems. In this note, it has been shown that the long-been-recognized sufficient condition is also necessary. This important result sheds new light on the nature of switched nonlinear systems and is worth the attention by the community.

Acknowledgment

We are grateful for the financial support of the Natural Science Foundation of Jiangsu Province (No. BK20140829), and the Fundamental Research Funds for the Central Universities (No. NS2016024).

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