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The Stability of High Order Max-Type Difference Equation

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Abstract: In this paper, we investigate the stability of following max-type difference equation $x_{n+1} = p + \sum_{i=1}^{t} a_i x_{n-m_i} \max\left\{\frac{b_1}{x_{n-n_1}}, \cdots, \frac{b_r}{x_{n-n_r}}\right\}$, where $0 \le m_1 < m_2 < \cdots < m_t$, $0 \le n_1 < n_2 < \cdots < n_r$, with $\{m_1, m_2, \cdots, m_t\} \cap \{n_1, n_2, \cdots, n_r\} = \phi$, $a_i > 0 \ (i = 1,2,\cdots t)$, $b_j > 0 \ (j = 1,2,\cdots r)$ and $p > \max\{b_1, \cdots, b_r\} \cdot \sum_{i=1}^{t} a_i$, the initial values are positive. By constructing a system of equations and binary function, we show the equation has a unique positive equilibrium solution, and the positive equilibrium solution is globally asymptotically stable. The conclusion of this paper extends and supplements the existing results.

Keywords: Difference Equations, Positive Solution, Convergence, Globally Stable

1. Introduction

In mathematics, recursive relation, which is difference equation, is a kind of recursion formula to define a sequence: the sequence of each item is defined as a function. Difference equation is the discretization of differential equations. The difference system is described the mathematical model of discrete system, it is an important branch of dynamical system, the application of its theory is rapidly broadening to various fields, such as economics, ecology, physics, engineering, control theory, computer science and so on (see [1-4]). The stability and global behavior is one of the hot spots in researches about difference equation model, the conclusion has a certain guiding role to production practices.

In recent years, more and more researches on the dynamic behaviors of higher order nonlinear difference equations have been studied (see [5-19]). One of the classes of such difference equations are max-type difference equations (see [10-19]).

In [16], Amleh studied the nonlinear difference equation $x_{n+1} = p + \frac{x_{n-1}}{x_n}$, showed that the unique positive equilibrium solution $\bar{x} = p + 1$ is globally asymptotically stable:

In [17], Fan studied the higher order difference equation $x_{n+1} = f(x_n, x_{n-k})$, and gave a sufficient condition for its global asymptotical stability, these results are applied to the difference equation $x_{n+1} = a + \frac{x_{n-k}}{x_n}$.

In [18], Sun studied global behavior of the max-type difference equation $x_{n+1} = \max\{1/x_{n-m}, A_n / x_{n-r}\}$, proved that if $A_n \in (0,1)$ and $\sup A_n < 1$ is a periodic sequence, then every positive solution of this equation is eventually periodic with period 2m.

In [19], Stević studied behavior of positive solutions of the following max-type system of difference equations,

$$x_{n+1} = \max\{c, \frac{y_n^p}{x_{n-1}^p}\}, \ y_{n+1} = \max\{c, \frac{x_n^p}{y_{n-1}^p}\},$$

proved that if $p, c \in (0,1)$ then every positive solution of the system converges to (1,1).

In this paper, we investigate the global stability of following max-type difference equation

$$x_{n+1} = p + \sum_{i=1}^{t} a_i x_{n-m_i} \max\left\{\frac{b_1}{x_{n-n_1}}, \cdots, \frac{b_r}{x_{n-n_r}}\right\} \tag{1}$$

where $n = 0,1,2,\cdots$, $0 \le m_1 < m_2 < \cdots < m_t$, $0 \le n_1 < n_2 < \cdots < n_r$ with $\{m_1, m_2, \cdots, m_t\} \cap \{n_1, n_2, \cdots, n_r\} = \varphi$, $a_i > 0$ $(i = 1,2,\cdots t)$, $b_j > 0$ $(j = 1,2,\cdots r)$ and $P > \max\{b_1, \cdots, b_r\} \cdot \sum_{i=1}^{t} a_i$, the initial values are positive. By constructing a system of equations and binary function, we will formulate and prove the equation has a unique positive equilibrium solution, and the positive equilibrium solution is globally asymptotically stable. The conclusion of this paper extends and supplements the existing results, this conclusion has a certain guiding role to production practices as a mathematical model.

For convenience, we denote $l = \max\{m_t, n_r\}$, $A = \sum_{i=1}^{t} a_i$, $B = \max\{b_1, \cdots, b_r\}$. So $p > AB$.

2. Some Definitions

In this section we will introduce some definitions (see [20]) which will be needed.

Definition A. [20] Let I be some interval of numbers and let $f : I \times I \to I$ be a continuously difference function.

A difference equation of order (k+1) is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}),\ n = 0,1,\cdots$$

A point $\bar{x} \in I$ is called equilibrium solution of the difference equation if $\bar{x} = f(\bar{x}, \bar{x}, \cdots, \bar{x})$, that is $x_n = \bar{x}$ for all $n \ge -k$.

Definition B. [20] The equilibrium $\bar{x}$ is called locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of difference equation with initial values satisfied $\left|x_{-k} - \bar{x}\right| + \left|x_{1-k} - \bar{x}\right| + \cdots + \left|x_0 - \bar{x}\right| < \delta$, then

$$\left|x_n - \bar{x}\right| < \varepsilon \text{ for all } n \ge -k.$$

Definition C. [20] The equilibrium $\bar{x}$ is called a global attractor if for every solution $\{x_n\}_{n=-k}^{\infty}$ of difference equation, we have $\lim_{n\to\infty} x_n = \bar{x}$.

Definition D. [20] The equilibrium $\bar{x}$ of difference equation is called globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of the difference equation.

3. Main Results

In this section we formulate and prove some lemmas and main theorems in this paper, obtain that every positive solution of (1) has to be the ultimate form of globally asymptotically stable.

Theorem 1. Equation (1) has a unique positive equilibrium solution $\bar{x} = p + AB$.

Proof. Since

$$\bar{x} = p + \sum_{i=1}^{t} a_i \bar{x} \max\left\{\frac{b_1}{\bar{x}}, \cdots, \frac{b_r}{\bar{x}}\right\}$$

we have $\bar{x} = p + A\bar{x} \cdot \frac{B}{\bar{x}} = p + AB > p$, so equation (1) has a unique positive equilibrium $\bar{x} = p + AB$. #

Equation $p + \sum_{i=1}^{t} a_i x \cdot \max\left\{\frac{b_1}{p}, \cdots, \frac{b_r}{p}\right\} = x$, that is $p + Ax \cdot \frac{B}{p} = x$, the only fixed point for the solution of this equation is $x = \frac{p^2}{p - AB}$, denote by D, that is $D = \frac{p^2}{p - AB}$. Because

$$D - p = \frac{p^2}{p - AB} - p = \frac{p^2 - p(p - AB)}{p - AB} = \frac{pAB}{p - AB} > 0$$

so $D > p$.

Lemma 1. For any real number $H \ge D$, if initial values $x_0, x_{-1}, \cdots, x_{-l} \in [p, H]$, then $x_n \in [p, H]$ $(n \ge 1)$.

Proo f. Since $H \ge D = \frac{p^2}{p - AB}$, so $H \ge \frac{p^2 + ABH}{p}$, for any $x_0, x_{-1}, \cdots, x_{-l} \in [p, H]$, we have

$$p \le x_1 = p + \sum_{i=1}^{t} a_i x_{-m_i} \max\left\{\frac{b_1}{x_{-n_1}}, \cdots, \frac{b_r}{x_{-n_r}}\right\}$$

$$\le p + \frac{ABH}{p} = \frac{p^2 + ABH}{p} \le H,$$

$$p \le x_2 = p + \sum_{i=1}^{t} a_i x_{1-m_i} \max\left\{\frac{b_1}{x_{1-n_1}}, \cdots, \frac{b_r}{x_{1-n_r}}\right\}$$

$$\le p + \frac{ABH}{p} = \frac{p^2 + ABH}{p} \le H,$$

suppose for every $n \le k$, there is $x_n \in [p, H]$, then

$$p \le x_{k+1} = p + \sum_{i=1}^{t} a_i x_{k-m_i} \max\left\{\frac{b_1}{x_{k-n_1}}, \cdots, \frac{b_r}{x_{k-n_r}}\right\}$$

$$\le p + \frac{ABH}{p} = \frac{p^2 + ABH}{p} \le H.$$

By induction, for every $n \ge 1$, we have $x_n \in [p, H]$. #

Let $h_0 = p$, $H_0 = H \ge D$, for any $i \ge 0$, define the system of equations as follows:

$$h_{i+1} = p + \sum_{i=1}^{t} a_i h_i \max\left\{\frac{b_1}{H_i}, \cdots, \frac{b_r}{H_i}\right\} \quad (2)$$

$$H_{i+1} = p + \sum_{i=1}^{t} a_i H_i \max\left\{\frac{b_1}{h_i}, \cdots, \frac{b_r}{h_i}\right\} \quad (3)$$

that is $h_{i+1} = p + Ah_i \cdot \frac{B}{H_i}$, $H_{i+1} = p + AH_i \cdot \frac{B}{h_i}$.

Lemma 2. For every $n \ge 0$, there is

$$h_n \le h_{n+1} < \bar{x} < H_{n+1} \le H_n$$

and $\lim_{n\to\infty} H_n = \lim_{n\to\infty} h_n = \bar{x}$.

Proof. Obtained by above definition (2-3), we have

$$h_0 \le h_1 = p + Ah_0 \cdot \frac{B}{H_0} = p + Ap \cdot \frac{B}{H} < p + AB = \bar{x}$$

$$H_0 \ge H_1 = p + AH_0 \cdot \frac{B}{h_0} = p + AH \cdot \frac{B}{p} > p + AB = \bar{x}$$

so $p = h_0 \le h_1 < \bar{x} < H_1 \le H_0 = H$. Because

$$h_1 = p + Ah_0 \cdot \frac{B}{H_0} \le h_2 = p + Ah_1 \cdot \frac{B}{H_1}$$

$$< p + A\bar{x} \cdot \frac{B}{\bar{x}} = p + AB = \bar{x},$$

$$H_1 = p + AH_0 \cdot \frac{B}{h_0} \ge H_2 = p + AH_1 \cdot \frac{B}{h_1}$$

$$> p + A\bar{x} \cdot \frac{B}{\bar{x}} = p + AB = \bar{x},$$

so $h_0 \le h_1 \le h_2 < \bar{x} < H_2 \le H_1 \le H_0$.

By induction, there is $h_n \le h_{n+1} < \bar{x} < H_{n+1} \le H_n$ for every $n \ge 0$. That is

$$p = h_0 \le h_1 \le h_2 \le \cdots \le h_n \le \cdots < \bar{x} <$$

$$\cdots < H_n \le \cdots \le H_2 \le H_1 \le H_0 = H.$$

According to the monotone bounded theorem, we know the limits of $\{h_n\}, \{H_n\}$ are existence. Let $\bar{h} = \lim_{n\to\infty} h_n$, $\bar{H} = \lim_{n\to\infty} H_n$. Take limits on both sides of (2-3), then

$$\bar{h} = p + \sum_{i=1}^{t} a_i \bar{h} \cdot \max\left\{\frac{b_1}{\bar{H}}, \cdots, \frac{b_r}{\bar{H}}\right\}$$

$$\bar{H} = p + \sum_{i=1}^{t} a_i \bar{H} \cdot \max\left\{\frac{b_1}{\bar{h}}, \cdots, \frac{b_r}{\bar{h}}\right\}$$

that is $\bar{h} = p + A\bar{h} \cdot \frac{B}{\bar{H}}$, $\bar{H} = p + A\bar{H} \cdot \frac{B}{\bar{h}}$, therefore $\overline{hH} = P\bar{H} + AB\bar{h} = P\bar{h} + AB\bar{H}$, so $(p - AB)(\bar{h} - \bar{H}) = 0$. Since $p > AB$, so $\bar{h} = \bar{H} = p + AB$, that is $\lim_{n\to\infty} H_n = \lim_{n\to\infty} h_n = \bar{x}$. #

Theorem 2. The unique equilibrium $\bar{x} = p + AB$ of equation (1) is locally stable.

Proof. Set $H = D = \frac{p^2}{p - AB}$, h_n and H_n as defined in Lemma 2. For every $\varepsilon > 0$ with $0 < \varepsilon < \min\{D - \bar{x}, \bar{x} - p\}$, according to Lemma 2 and local boundedness, there exists $n \ge 0$ such that $\bar{x} - \varepsilon < h_n < \bar{x} < H_n < \bar{x} + \varepsilon$.

Take $0 < \delta = \min\{\bar{x} - h_n, H_n - \bar{x}\}$, that is $(\bar{x} - \delta, \bar{x} + \delta) \subset [h_n, H_n]$. Then for every $x_0, x_{-1}, \cdots, x_{-l} \in (\bar{x} - \delta, \bar{x} + \delta)$, we have

$$x_1 = p + \sum_{i=1}^{t} a_i x_{-m_i} \max\left\{\frac{b_1}{x_{-n_1}}, \cdots, \frac{b_r}{x_{-n_r}}\right\}$$

$$\le p + \sum_{i=1}^{t} a_i H_n \max\left\{\frac{b_1}{h_n}, \cdots, \frac{b_r}{h_n}\right\} = H_{n+1} \le H_n,$$

$$x_1 \ge p + \sum_{i=1}^{t} a_i h_n \max\left\{\frac{b_1}{H_n}, \cdots, \frac{b_r}{H_n}\right\} = h_{n+1} \ge h_n,$$

that is $x_1 \in [h_n, H_n] \subset (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$.

Similarly, by induction, there is $x_n \in [h_n, H_n] \subset (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$ for every $n \ge -l$. According to theorem B, the equilibrium $\bar{x} = p + AB$ is locally stable. #

Theorem 3. The unique equilibrium solution $\bar{x} = p + AB$

of equation (1) is globally asymptotically stable.

Proof. In Theorem 2, we have proved $\bar{x} = p + AB$ is locally stable, then we will prove $\bar{x} = p + AB$ is global attractor.

Set $H = \max\{x_1, \cdots, x_{l+1}, D\}$, h_n and H_n as defined in Lemma 2. Following Lemma 1, for every $n \ge 1$, there is $x_n \in [h_0, H_0] = [p, H]$. So

$$h_1 = p + Ah_0 \cdot \frac{B}{H_0} \le x_{(l+1)+1}$$

$$= p + \sum_{i=1}^{t} a_i x_{l+1-m_i} \max\left\{\frac{b_1}{x_{l+1-n_1}}, \cdots, \frac{b_r}{x_{l+1-n_r}}\right\}$$

$$\le p + AH_0 \cdot \frac{B}{h_0} = H_1,$$

$$h_1 = p + Ah_0 \cdot \frac{B}{H_0} \le x_{(l+2)+1}$$

$$= p + \sum_{i=1}^{t} a_i x_{l+2-m_i} \max\left\{\frac{b_1}{x_{l+2-n_1}}, \cdots, \frac{b_r}{x_{l+2-n_r}}\right\}$$

$$\le p + AH_0 \cdot \frac{B}{h_0} = H_1.$$

By induction, there is $x_n \in [h_1, H_1]$ for every $n \ge (l+1)+1$.

Similarly, we have $x_n \in [h_1, H_1]$ for every $n \ge 2(l+1)+1$. By induction, $x_n \in [h_k, H_k]$ for every $n \ge k(l+1)+1$ where $k = 0, 1, \cdots$.

Following Lemma 2, we know $\lim_{n\to\infty} H_n = \lim_{n\to\infty} h_n = \bar{x}$, so $\lim_{n\to\infty} x_n = \bar{x}$. By Definition C, we know $\bar{x} = p + AB$ is global attractor.

According to Definition D, it is obviously that the equilibrium $\bar{x} = p + AB$ of equation (1) is globally asymptotically stable. #

4. Example

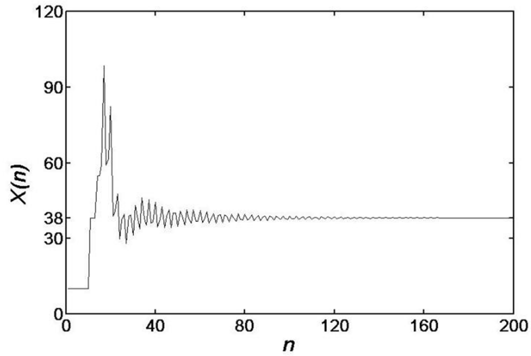
Consider one of example of differential equation (1):

$$x_{n+1} = 20 + (2x_{n-2} + 0.5x_{n-4} + 3.5x_{n-5}) \cdot \max\left\{\frac{0.5}{x_{n-3}}, \frac{3}{x_{n-6}}, \frac{1.5}{x_{n-9}}\right\} \quad n = 0, 1, \cdots \tag{4}$$

where the initial values $x_0, x_{-1}, \cdots, x_{-9} \in (0, +\infty)$. Obviously, it satisfies the conditions of Theorem 3, so the unique equilibrium $\bar{x} = 38$ of equation (4) is globally asymptotically stable. By giving the initial value assignment, the following figures 1-2 show the global asymptotic stability.

If initial values $x_0 = x_{-1} = \cdots = x_{-9} = 10$, equilibrium $\bar{x} = 38$ is globally asymptotically stable (see Figure 1).

If initialvalues $x_{-3} = x_{-6} = x_{-9} = 0.05$, $x_0 = x_{-1} = x_{-2} = x_{-4} = x_{-5} = x_{-7} = x_{-8} = 10$, equilibrium $\bar{x} = 38$ is globally asymptotically (see Figure 2).



Figures 1. *The solution of equation (4), when initial values* $x_0 = x_{-1} = \cdots = x_{-9} = 10$.

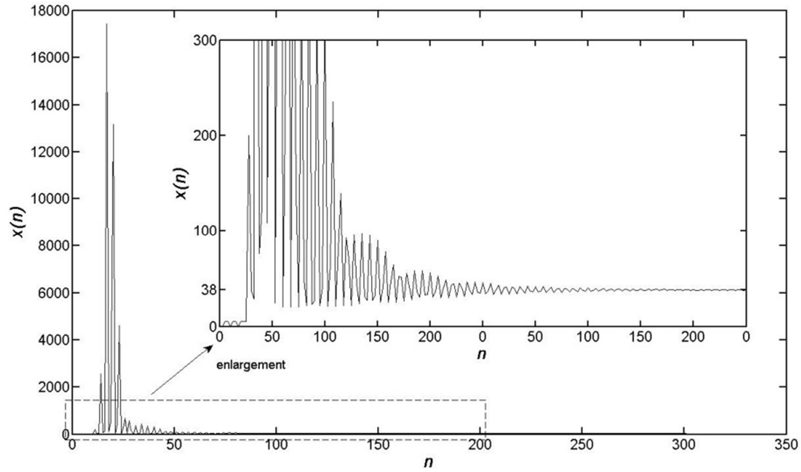


Figure 2. *The solution of equation (4), when initial values* $x_{-3} = x_{-6} = x_{-9} = 0.05$, $x_0 = x_{-1} = x_{-2} = x_{-4} = x_{-5} = x_{-7} = x_{-8} = 10$.

5. Conclusion

In this paper, we investigate the characters of positive solution of the max-type difference equation (1).

First, we showed equation (1) has unique positive equilibrium $\bar{x} = p + AB$.

Then, we proved two useful lemmas. By citing lemmas we showed the main theorems in this paper, that is the equilibrium solution $\bar{x} = p + AB$ of equation (1) is globally asymptotically stable.

At last, we give an example of difference equation (1), draw the trajectory of the solution by giving two different initial values, thus intuitively reflect the global asymptotic stability.

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