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Quenching for a Diffusion System with Coupled Boundary Fluxes

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Abstract: In this paper, we investigate a diffusion system of two parabolic equations with more general singular coupled boundary fluxes. Within proper conditions, we prove that the finite quenching phenomenon happens to the system. And we also obtain that the quenching is non-simultaneous and the corresponding quenching rate of solutions. This extends the original work by previous authors for a heat system with coupled boundary fluxes subject to non-homogeneous Neumann boundary conditions.

Keywords: Quenching, Quenching Rate, Quenching Point, Singular Term, Parabolic System

1. Introduction

In the present work, we mainly deal with the following diffusion system with singular coupled boundary fluxes

$$\begin{array}{ll} u_t(x,t)=u_{xx},\ \ v_t(x,t)=v_{xx}, & (x,t)\in(0,1)\times(0,T), \\ u_x(0,t)=f(v(0,t)),\ \ u_x(1,t)=0, & t\in(0,T), \\ v_x(0,t)=g(u(0,t)),\ \ v_x(1,t)=0, & t\in(0,T), \\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in[0,1]. \end{array} \tag{1}$$

For functions $u_0(x)$ and $v_0(x)$, we always assume that initial data satisfies $u_0'(x), v_0'(x)\ge 0$ and $u_0''(x),\ v_0''(x)\le 0$ with $0<u_0(x), v_0(x)<1$. To facilitate the following research, we also suppose that functions $f(v)$ and $g(u)$ verify the assumptions:

(H_1) $f(v)$ and $g(u)$ are locally Lipschitz on $u,v\in(0,1]$;

(H_2) $f'(v)<0$ and $g'(u)<0$ for $u,v\in(0,1]$;

(H_3) $\lim_{v\to 0^+} f(v)=\infty$ and $\lim_{u\to 0^+} g(u)=\infty$.

In the model (1), u and v can be thought as the temperatures of two mixed media during the heat propagation. This is a one-dimensional heat conduction rod of length 1 with positive initial temperatures $u_0(x)$, $v_0(x)$. At the left end {x=0}, heat is taken away with a rate $f(v(0,t))$ and $g(u(0,t))$ for u and v, respectively. The right end {x=1} is thermal isolation with $u_x(1,t)=v_x(1,t)=0$. Since the assumption proposed for the system implies that the two components are coupled completely and enhanced each other in the model. It is known that the singular negative flux at the boundary {x=0} may result in the so called finite time quenching of solutions, which makes it so interesting to investigate the quenching phenomenon of the solutions, see [2-5, 7-8, 11-13] and some survey papers [1, 6, 10]. Right here, we say that the solution (u,v) of the problem (1) quenches, if (u,v) exists in the classical sense and is positive for all $0\le t<T$ and satisfies

$$\liminf_{t\to T^-}\ \min_{0\le x\le 1}\{u(x,t),v(x,t)\}=0.$$

If this happens, T will be called as quenching time. Since a singularity develops in the absorption term at quenching time T, thus the classical solution doesn't exist anymore.

Due to the great work by many previous researchers, the blow-up problems of parabolic equation have been studied gradually matured, thus plenty of authors have begun to pay attention to the quenching phenomena and become a heated study field.

Ferreira, Pablo and Quirs. etc in [2] studied a system of heat equations coupled at the boundary

$$\begin{array}{ll} u_t(x,t)=u_{xx},\ \ v_t(x,t)=v_{xx}, & (x,t)\in(0,1)\times(0,T), \\ u_x(0,t)=v^{-p}(0,t),\ \ u_x(1,t)=0, & t\in(0,T), \\ v_x(0,t)=u^{-q}(0,t),\ \ v_x(1,t)=0, & t\in(0,T), \\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in[0,1]. \end{array} \tag{2}$$

They obtained that if $p,q\ge 1$, and then quenching is always simultaneous. While if $p<1$ or $q<1$, non-simultaneous quenching indeed occurs. If $0<p,q<1$, then there exists initial data such that simultaneous quenching produces. Besides, if quenching is non-simultaneous and, for instance u is the quenching variable, then $u(0,t)\sim(T-t)^{\frac{1}{q+1}}$ and $u(0,T)\sim x$, where and throughout this paper, the notation $f\sim g$ means that there exist two positive constants c_1,c_2 such that $c_1 f\le g\le c_2 f$ with some $c_1\le c_2$ holds for t close to the quenching time T.

Zheng and Song in [3] studied phenomena of non-simultaneous quenching to a coupled heat system

$$\begin{array}{ll} u_t(x,t)=u_{xx},\ \ v_t(x,t)=v_{xx}, & (x,t)\in(0,1)\times(0,T), \\ u_x(0,t)=0,\ \ u_x(1,t)=-v^{-p}(0,t), & t\in(0,T), \\ v_x(0,t)=0,\ \ v_x(1,t)=-u^{-q}(0,t), & t\in(0,T), \\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in[0,1]. \end{array} \tag{3}$$

They gave an accurate non-simultaneous quenching classification and the corresponding quenching rates of (1.3) were determined as below:

$$u(1,t)\sim(T-t)^{\frac{p-1}{2(pq-1)}},\ v(1,t)\sim(T-t)^{\frac{q-1}{2(pq-1)}},\ \text{if}\ \ p,q>1\ \text{ or }\ p,q<1;$$
$$u(1,t)\sim(T-t)^{\frac{1}{4}},\ v(1,t)\sim(T-t)^{\frac{1}{4}},\ \text{if}\ \ p=q=1;$$
$$u(1,t)\sim(T-t)^{\frac{p-1}{2(pq-1)}},\ v(1,t)\sim(T-t)^{\frac{q-1}{2(pq-1)}},\ \text{if}\ \ q>p=1;$$

for simultaneous quenching, and $u(1,t)\sim(T-t)^{\frac{1}{q+1}}$ for non-simultaneous quenching.

Fila and Levine in [4] studied the following finite time quenching for the scalar equations

$$\begin{array}{ll} u_t(x,t)=u_{xx}, & (x,t)\in(0,1)\times(0,T), \\ u_x(0,t)=0,\ \ u_x(1,t)=-v^{-q}(1,t), t\in(0,T), & \\ u(x,0)=u_0(x), & x\in[0,1]. \end{array} \tag{4}$$

They obtained the quenching rate is $u(1,t)\sim(T-t)^{\frac{1}{2(q+1)}}$ as $t\to T^-$.

Ji, Qu and Wang in [5] considered finite time quenching problem for parabolic system

$$\begin{array}{ll} u_t(x,t)=u_{xx},\ \ v_t(x,t)=v_{xx}, & (x,t)\in(0,1)\times(0,T), \\ u_x(0,t)=(u^{-m}+v^{-p})(0,t),\ \ u_x(1,t)=0, & t\in(0,T), \\ v_x(0,t)=(u^{-q}+v^{-n})(0,t),\ \ v_x(1,t)=0, & t\in(0,T), \\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in[0,1], \end{array} \tag{5}$$

where $m,n\ge 0, p,q>0$, $u_0(x)$ and $v_0(x)$ are smooth positive initial data. They obtained that if v does not quench in (5), then $q<m+1$. If $q\ge m+1, p\ge n+1$, then any quenching in (5) must be simultaneous, while if $p<n+1$, then there exist initial data such that v quenches but u doesn't. If $q\le\frac{n(m+1)}{n+1}\left(p\le\frac{m(n+1)}{m+1}\right)$, and $p\ge n+1$ $(q\ge m+1)$ then the component u(v) quenches alone under any positive initial data. Besides, if $p<n+1$ and $q<m+1$, then both simul -taneous and non-simultaneous quenching may occur in (5), which depends on the initial data. And the set of initial data such that one component quenches alone is open. Furthermore, assume that u quenches at time T with v keeping positive in (5), then $u(0,t)\sim(T-t)^{\frac{1}{2m+2}}$. On the other hand, simultaneous quenching rates are also discussed under different conditions.

Some authors also studied the following coupled heat equations with nonlinear terms. For example, A de Pablo, F. Quirós and J. D. Rossi [6] studied the non-simultaneous quenching in a semilinear parabolic system

$$\begin{array}{ll} u_t(x,t)=u_{xx}-v^{-p},\ \ v_t(x,t)=v_{xx}-u^{-q}, & (x,t)\in(0,1)\times(0,T), \\ u_x(0,t)=u_x(1,t)=0, & t\in(0,T), \\ v_x(0,t)=v_x(1,t)=0, & t\in(0,T), \\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in[0,1], \end{array} \tag{6}$$

Zhi and Mu in [7] studied the non-simultaneous quenching in a semilinear parabolic system

$$\begin{aligned} & u_t(x,t) = u_{xx} + \log(\alpha v), \quad v_t(x,t) = v_{xx} + \log(\beta u), && (x,t) \in (0,1)\times(0,T), \\ & u_x(0,t) = u_x(1,t) = 0, && t \in (0,T), \\ & v_x(0,t) = v_x(1,t) = 0, && t \in (0,T), \\ & u(x,0) = u_0(x), \quad v(x,0) = v_0(x), && x \in [0,1]. \end{aligned} \tag{7}$$

And Ji, Zhou and Zheng in [8] studied the coupled system

$$\begin{aligned} & u_t(x,t) = u_{xx} - (u^{-m} + v^{-p}), && (x,t) \in (0,1)\times(0,T), \\ & v_t(x,t) = v_{xx} - (u^{-q} + v^{-n}), && (x,t) \in (0,1)\times(0,T), \\ & u_x(0,t) = u_x(1,t) = 0, && t \in (0,T), \\ & v_x(0,t) = v_x(1,t) = 0, && t \in (0,T), \\ & u(x,0) = u_0(x), \quad v(x,0) = v_0(x), && x \in [0,1]. \end{aligned} \tag{8}$$

All of them have identified simultaneous and non-simultaneous quenching by a precise classification of parameters, and establish simultaneous quenching rates or non-simultaneous quenching rates.

Motivated by those papers and references therein, the main purpose of this paper is to study a more general system (1) to obtain a lot of more general conclusions for the non-simultaneous quenching phenomenon with coupled fluxes at the boundary, which appeared in many papers with some special case, see [2-6, 11-13].

2. Main Results and Proof

In this section, we mainly deal with the non-simultaneous quenching, quenching rates and quenching set.

At first, we will prove a priori estimate to begin our study, which ensures that quenching always happens for the diffusion system (1.1). To simplify the presentation of the proofs, we define the functions as follows

$$\Phi(t) = u(0,t) = \min_{0\le x\le 1} u(x,t), \quad \Psi(t) = v(0,t) = \min_{0\le x\le 1} v(x,t). \tag{9}$$

Lemma 2.1 Quenching happens for system (1.1) for every initial data.

Proof: By the maximum principle we have

$$u \le M = \|u_0\|_\infty, \quad v \le N = \|v_0\|_\infty .$$

Therefore, by integrating $(1)_1$ in the interval [0, 1], we can obtain

$$\int_0^t\int_0^1 u_s \mathrm{d}x\mathrm{d}s = \int_0^t\int_0^1 u_{xx}\mathrm{d}s\mathrm{d}x = \int_0^t -f(v(0,s))\mathrm{d}s.$$

Since $f(v)$ is locally Lipschitz on (0, 1] and $f'(v) < 0$ for $v \in (0,1]$, hence we have

$$\int_0^1 (u(x,t) - u_0(x))\mathrm{d}x \le -tf(N).$$

This implies the following mass estimates,

$$0 < \int_0^1 u(x,t)\mathrm{d}x \le M - tf(N).$$

Similarly, by integrating $(1)_1$ in the interval [0, 1], we can obtain

$$\int_0^t\int_0^1 v_s \mathrm{d}x\mathrm{d}s = \int_0^t\int_0^1 v_{xx}\mathrm{d}s\mathrm{d}x = \int_0^t -g(u(0,s))\mathrm{d}s.$$

Since $g(u)$ is locally Lipschitz on (0, 1] and $g'(u) < 0$ for $u \in (0,1]$, hence we have

$$\int_0^1 (v(x,t) - v_0(x))\mathrm{d}s\mathrm{d}x \le -tg(M),$$

Thus we can also get the following mass estimates,

$$0 < \int_0^1 v(x,t)\mathrm{d}x \le N - tg(M).$$

Consequently, there exists a finite time T, such that quenching happens as $t \to T$. Otherwise it will produce a contradictions if u, v are positive for all times.

Lemma 2.2 There exists a positive constant δ > 0, such that

$$\Phi'(t) \le -\delta g(\Phi(t)), \quad \Psi'(t) \le -\delta f(\Psi(t)), \quad t \in [0,T). \tag{10}$$

Proof: Consider functions $F = u_t + \delta v_x$, $G = v_t + \delta u_x$, it's easy to check that F, G are solutions to the heat equation. If we choose $\delta > 0$ small enough, for every $x \in [0,1]$, we have $F(x,0), G(x,0) < 0$. Notice that u and v are decreasing in time, so we can get $F(1,t), G(1,t) < 0$. As to the flux at $x = 0$, we have

$$F_x = (f'(v) + \delta)v_t \ge -F(0,t), \quad G_x = (g'(v) + \delta)u_t \ge -G(0,t)$$

with δ small sufficiently. Thus by the maximum principle, we can obtain that $F(x,t), G(x,t) \le 0$ for every $x \in [0,1]$ and $t \in [0,T)$. The result in (2.5) is just the particular case for $x = 0$.

Moreover, we have the following estimates via directly integrating for inequalities (2.5).

Corollary 2.1 $\int_0^{\Phi(t)} \frac{\mathrm{d}\tau}{g(\tau)} \ge C(T-t)$, $\int_0^{\Psi(t)} \frac{\mathrm{d}\xi}{f(\xi)} \ge C(T-t)$.

Within these estimates we can obtain the following corollary.

Corollary 2.2 The quenching time is continuous with respect to the initial data.

Since the proof is similar to the Theorem 2.1 in [1], we omit here.

Lemma 2.3 There exists a constant $C > 0$ such that,

$$\Phi'(t) \geq -C\frac{\mathrm{d}f(\Psi)}{\mathrm{d}\Psi}g(\Phi),\ \Psi'(t) \geq -C\frac{\mathrm{d}g(\Phi)}{\mathrm{d}\Phi}f(\Psi). \quad (11)$$

Proof: Let $H = u_x - \varphi(x)f(v)$, $I = v_x - \varphi(x)g(u)$, where $\varphi:[0,1]\to[0,1]$ is a nonnegative, non-increasing, convex C^2 function such that $\varphi(0)=1, \varphi(1)=0$, and $\varphi(x) \leq u_0'(x)f(v_0(x)), \varphi(x) \leq v_0'(x)g(u_0(x))$, for $x \in [0,1]$. It's easy to find that H and I are nonnegative at $t = 0$. Besides, differentiating H we can get

$$H_t - H_{xx} = \varphi''(x)f(v) + 2\varphi'(x)f'(v)v_x + \varphi(x)f''(v)(v_x)^2 \geq 0.$$

Similarly, we can get

$$I_t - I_{xx} = \varphi''(x)g(u) + 2\varphi'(x)g'(u)u_x + \varphi(x)g''(u)(u_x)^2 \geq 0.$$

In other words, H and I are super-solutions for the heat equation. In addition, they vanish at the border $x = 0$ and $x = 1$. Hence $H(x,t), I(x,t) \geq 0$ for every $(x,t) \in [0,1]\times[0,T)$, which implies $H_x, I_x \geq 0$ for some particular case, that is to say,

$$u_t(0,t) = u_{xx}(0,t) \geq \varphi'(0)f(v(0,t)) + f'(v(0,t))v_x(0,t) \geq -Cf'(v(0,t))g(u(0,t)).$$

And the analogous estimate holds for v,

$$v_t(0,t) = v_{xx}(0,t) \geq \varphi'(0)g(u(0,t)) + g'(u(0,t))u_x(0,t) \geq -Cg'(u(0,t))f(v(0,t)).$$

To this end, the proof of this lemma is complete.

Lemma 2.4 The quenching point is only the origin $x = 0$.

Proof: Since $H(x,t) \geq 0$, we have $u_x(x,t) \geq \varphi(x)f(v(x,t)) \geq \frac{f(N)}{3}$ for every such that $\varphi(x_0) = \frac{1}{3}$. Therefore, we obtain $u(x,t) \geq u(0,t) + Cx$. The similar estimate also holds for v. Thus we can obtain that the quenching point is only the origin.

Theorem 2.1 Let (u_t, v_t) be time-derivatives of (u,v), and then (u_t, v_t) will blow up at quenching point $x = 0$ simultaneously.

Proof: Define functions $J_1(x,t) = u_t + \varepsilon g(v(0,t)), J_2(x,t) = v_t + \varepsilon f(u(0,t)), 0 \leq x \leq 1, \tau \leq t < T,$ where $\tau \in (0,T)$, ε is some positive constant. Thus $J_1(x,t)$ and $J_2(x,t)$ verify the follows,

$$(J_1)_t - (J_1)_{xx} = \varepsilon g'(v(0,t))v_t(0,t) \geq 0,$$
$$(J_2)_t - (J_2)_{xx} = \varepsilon f'(u(0,t))u_t(0,t) \geq 0.$$

So we have $J_1(x,\tau) > 0, J_2(x,\tau) > 0$. Furthermore, we can obtain

$$(J_1)_x(1,t) = (J_2)_x(1,t) = 0. \quad \tau < t < T,$$
$$(J_1)_x(0,t) \geq 0, (J_2)_x(0,t) \geq 0. \quad \tau < t < T.$$

Since $u_t(x,0) < 0$ and $g(v(0,t))$ is bounded, select a proper ε, we have $J_1(x,0) \geq 0$.

Similarly, we also have $J_2(x,0) \geq 0$. By the maximum principle, we can get $J_1(x,t) \geq 0, J_1(x,t) \geq 0$, which means

$$u_t \geq -\varepsilon g(v(0,t)), 0 \leq x \leq 1, \tau \leq t < T,$$
$$v_t \geq -\varepsilon f(u(0,t)), 0 \leq x \leq 1, \tau \leq t < T.$$

Let $t \to T^-$, we can get the conclusion

$$\lim_{t\to T^{-1}} u_t = \infty, \lim_{t\to T^{-1}} v_t = \infty.$$

To this end, the proof is complete.

Theorem 2.2 If quenching is non-simultaneous and let u be the quenching variable, then

$$u(0,t) \sim g^{-1}(C(T-t)),\ u_t(0,t) \sim C(T-t) \text{ and } u(x,T) \sim x.$$

Proof: In Lemma 2.1, we have given the lower bound of the non-simultaneous rate, while the upper bound can be obtained easily by integrating the first estimate in (11). Using that $\Psi \geq$ C>0 : $g(\Phi(t)) \leq C(T-t)$. As $x \to 0$, by lower estimate given in Corollary 2.1, then upper estimate follows directly from the fact that u is concave; therefore

$$u_x(x,t) \leq u_x(0,t) = f(v(0,t)) \leq C.$$

To this end, the proof of Theorem 2.1 is complete.

3. Conclusion

Throughout this paper, we have studied the solutions of a parabolic system of heat equations coupled at the boundary through a singular flux. This system displays a singularity in finite time, which is called quenching in the literature. We obtained the quenching point is the origin, non-simultaneous quenching rates. To some degree, our work extends the original work by previous authors for a heat system with coupled boundary fluxes for a more general boundary flux.

We have to admit that there are still many possible improvements and extensions of our results. One possibility is that we consider the diffusion process in a higher dimension. If we study the radial solutions in a ball, some similar results may hold as well. Besides, we can extend the local diffusion to nonlocal diffusion, which may be more effective to describe the real situation. Another aspect for us to improve is to find a method to identify the non-simultaneous quenching and simultaneous quenching, which once was determined by some parameters.

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References

[1] H. Kawarada. On solutions of initial boundary value problem for $u_t = u_{xx} + \frac{1}{1-u}$, Publ. Res. Inst. Math. Sci. 10(1975), 729-736.

[2] R. Ferreira, A de Pablo, F. Quirs, J. D. Rossi. Non-simultaneous quenching in a system of heat equations coupled at the boundary. Z. Angew. Math. Phys., 57(2006), 586-594.

[3] S. N. Zheng, X. F. Song. Quenching rates for heat equations with coupled nonlinear boundary flux. Sci. China Ser. A. 2008; 51: 1631-1643.

[4] M. Fila, H. A. Levine. Quenching on the boundary. Nonlinear Anal., 21(1993), 795-802.

[5] R. H. Ji, C. Y. Qu, L. D. Wang. Simultaneous and non-simultaneous quenching for coupled parabolic system, Appl. Anal., 94(2), 2015, 233-250.

[6] A. de Pablo, F. Quirós, J. D. Rossi. Non-simultaneous quenching, Appl. Math. Lett. 15 (2002), 265–269.

[7] Y. H. Zhi, C. L. Mu, Non-simultaneous quenching in a semilinear parabolic system with weak singularities of logarithmic type. Applied Mathematics and Computation, 196(2008), 17-23.

[8] R. H. Ji, S. S. Zhou, S. N. Zheng. Quenching behavior of solutions in coupled heat equations with singular multi-nonlinearities, Applied Mathematics and Computation, 223 (2013), 401–410.

[9] C. Y. Chan. Recent advances in quenching phenomena, Proc. Dynam. Systems. Appl. 2(1996), 107-113.

[10] H. A. Levine. The phenomenon of quenching: a survey, in: Trends in the Theory and Practice of NonLinear Analysis, North Holland, New York, 1985, pp. 275-286.

[11] H. A. Levine, J. T. Montgomery. The quenching of solutions of some nonlinear parabolic equations, SIAM J. Math. Anal. 11 (1980), 842-847.

[12] T. Salin. On quenching with logarithmic singularity, Nonlinear Anal. TMA. 52 (2003), 261-289.

[13] C. L. Mu, S. M. Zhou, D. M. Liu. Quenching for a reaction-diffusion system with logarithmic singularity. Nonlinear Anal., 71(2009), 5599-5605.