AppliedandComputationalMathematics
2014; 3(2): 43-47
Published online March 20, 2014 (http://www.sciencepublishinggroup.com/j/acm)
doi: 10.11648/j.acm.20140302.11

On optimization of a Coxian queueing model with two phases

Vedat Sağlam[1], Merve Uğurlu[1], Erdinç Yücesoy[1], Müjgan Zobu[2], Murat Sağır[1]

[1]Department of Statistics, Faculty of Science and Arts, OndokuzMayıs University, Kurupelit, Turkey
[2]Department of Statistics, Faculty of Science and Arts, Amasya University, Amasya, Turkey

Email address:
mujganzobu@hotmail.com (M. Zobu)

To cite this article:
Vedat Sağlam, Merve Uğurlu, Erdinç Yücesoy, Müjgan Zobu, Murat Sağır. On Optimization of a Coxian Queueing Model with Two Phases. *AppliedandComputationalMathematics*. Vol. 3, No. 2, 2014, pp. 43-47. doi: 10.11648/j.acm.20140302.11

Abstract: In this study we have obtained stochastic equation systems of a Coxian queueing model with two phases where arrival stream of this model is according to the exponential distribution with λ parameter. The service time of any customer at server i $(i = 1,2)$ is exponential with parameter μ_i. In addition we have obtained state probabilities of this queueing model at any given t moment.Furthermore performance measures of this queueing system are calculated. Various queueing systems are found for some values of α probability and service parameters: if $\alpha = 1$and $\mu_1 = \mu_2$taken then $M/E_2/1/\ 0$ queueing model is obtained, for $\alpha = 1$it is shown that service time of a customer is according to hypoexponential, if $\alpha = 0$ is taken we have $M/\ M/1/\ 0$ queueing system. Lately,an application of this queueing model is done. The optimal value of the mean customer number in the system is found. Finally, optimal ordering according to the loss probability is obtained by changing the service parameters .A numerical example is given on the subject

Keywords: Coxian Model, Differential Equations, Loss Probability, Limiting Distribution, Optimal Ordering

1. Introduction

The queueing systems with phase-type are one of the important part of the queueing theory. It is possible to construct phase-type distributions that are a mixture of hypoexponential and hyperexponential distributions to obtain increasingly complex representations. With the introduction of the mathematically expedient notion of "complex probabilities" and "complex rates," [1] showed how any distribution having a rational Laplace transform could be represented by a sequence of exponential phases. The sequence of phase could be arranged one after the other in series formation, with the provision of permitting termination after the completion of any phase, explained in [2]. Another large subset of Coxian distributions is the phase-type distribution introduced by [3], which can be considered to be a natural probabilistic generalization of the Erlang, given in [4]. In [5], equilibrium state distributions were determined for queues with load-dependent Poisson arrivals and service time distributions representable by Cox's generalized method of stages. The solution was obtained by identifying a birth-death process that has the same equilibrium state distribution as the original queue. In [6], the optimal ordering of the tandem server with two stage is given. In [7], Optimal sample size is obtained depending on the probabilities of type 1 and type 2 errors in a queueing system with two channel in which service time has Coxian distribution and arrivals are Poisson distribution. In this paper we have obtained steady state equation systems of a Coxian queueing model with two phases where the arrivals to this model are according to the exponential distribution with λ parameter . We have obtained state probabilities of the system with respect to α at any given t moment. The optimal values of loss probability and the mean customer number in the system are found. Furthermore, loss probability by changing the service parameters are obtained for optimal ordering of phases. In this system if $\alpha = 1$ and $\mu_1 = \mu_2$taken then $M/E_2/1/\ 0$ queueing model is obtained. Also for $\alpha = 1$ it is shown that service time of a customer is according to hypoexponential. If $\alpha = 0$ is taken we have $M/\ M/1/\ 0$ queueing system. In addition an application of this queueing model is given

2. Stochastic Model

We have obtained stochastic equation systems of a Coxian queueing model with two phases where the incomes to this model are according to the exponential distribution with λ parameter . The service time of any customer at phases i ($i = 1,2$) is exponential with parameter μ_i. First phase and second phase can be empty or busy but in this system it is not allowed to be two customers in a phase at the same time. Let ξ_t be the state of first phase and η_t be the state of second phase at any t moment. We have obtained the state probabilities with respect to α at any given t moment. Limit probabilities, differential and difference equations of this model given later.

2.1. Limit Distribution

Here $\{(\xi_t, \eta_t), t \geq 0\}$ is a 2 - dimensional Markov chain with continuous parameter and state space
$\Omega = \{(0,0), (0,1), (1,0))\}$.

$$P_{ij}(t) = \text{Prob}\{\xi_t = i, \eta_t = j\}. \forall (i,j) \in \Omega \tag{1}$$

Kolmogorov differential equation for these probabilities is obtained. The transient probabilities of the process $\{(\xi_t, \eta_t), t \geq 0\}$ is found for $(t, t+h)$, namely

$$P_{00}(t+h) = \big(1 - \lambda h + o(h)\big)P_{00}(t) + (1-\alpha)\big(\mu_1 h + o(h)\big)P_{10}(t) + \big(\mu_2 h + 0(h)\big)P_{01}(t) + o(h) \tag{2}$$

$$P_{01}(t+h) = \big(1 - \mu_2 h + o(h)\big)P_{01}(t) + \alpha\big(\mu_1 h + o(h)\big)P_{10}(t) + o(h) \tag{3}$$

$$P_{10}(t+h) = \big(1 - \mu_1 h + o(h)\big)P_{10}(t) + \big(\lambda h + o(h)\big)P_{00}(t) + o(h) \tag{4}$$

We write (2), (3) and (4) equations as follows as $h \to 0$,

$$P'_{00}(t) = -\lambda P_{00}(t) + (1-\alpha)\mu_1 P_{10}(t) + \mu_2 P_{01}(t) \tag{5}$$

$$P'_{01}(t) = -\mu_2 P_{01}(t) + \alpha\mu_1 P_{10}(t) \tag{6}$$

$$P'_{10}(t) = -\mu_1 P_{10}(t) + \lambda P_{00}(t) \tag{7}$$

It is supposed that limiting distribution of $P_{ij}(t)$ are exist as followings:

$$\lim_{t\to\infty} P_{ij}(t) = \pi_{ij} \tag{8}$$

$$\lim_{t\to\infty} P'_{ij}(t) = 0 \tag{9}$$

Limiting distribution has been widely given in [8].

Steady- state equations for $\{(\xi_t, \eta_t), t \geq 0\}$ are obtained as following:

$$0 = -\lambda\pi_{00} + (1-\alpha)\mu_1\pi_{10} + \mu_2\pi_{01} \tag{10}$$

$$0 = -\mu_2\pi_{01} + \alpha\mu_1\pi_{10} \tag{11}$$

$$0 = -\mu_1\pi_{10} + \lambda\pi_{00} \tag{12}$$

$$\sum\sum_{(i,j)\epsilon\Omega} \pi_{ij} = 1 \tag{13}$$

We define $\rho_1 = \lambda/\mu_1$ and $\rho_2 = \lambda/\mu_2$. If the equations (10), (11) and (12) are solved, the following transient probabilities are obtained:

$$\pi_{01} = \alpha\rho_2\pi_{00} \tag{14}$$

$$\pi_{10} = \rho_1\pi_{00} \tag{15}$$

If the obtained transient probabilities and $\pi_{00}$are put in the equation (13),

$$\pi_{00}(1 + \rho_1 + \alpha\rho_2) = 1 \tag{16}$$

$$\pi_{00} = \frac{1}{1 + \rho_1 + \alpha\rho_2} \tag{17}$$

is found. π_{01} and π_{10} probabilities are calculated if the equation (17) is put in (14) and (15).

2.2. Probability Function

We define $\rho_1 = \lambda/\mu_1$ and $\rho_2 = \lambda/\mu_2$. If we shall solve (10), (11) and (12) equations under condition (13), we obtain the following two dimension probability function:

$$\pi_{ij} = \begin{cases} \dfrac{1}{1+\rho_1+\rho_2\alpha}, & (i,j) = (0,0) \\ \dfrac{\rho_2\alpha}{1+\rho_1+\rho_2\alpha}, & (i,j) = (0,1) \\ \dfrac{\rho_1}{1+\rho_1+\rho_2\alpha}, & (i,j) = (1,0) \\ 0, & otherwise. \end{cases} \tag{18}$$

Let π_0 be the probability of being no customer in system and π_1 to be the probability of being one customer in the system. Where,

$$\pi_0 = \pi_{00}, \quad \pi_1 = \pi_{01} + \pi_{10} \tag{19}$$

3. Obtaining the Measures of Performance

3.1. Coxian Queue Using z-Transform

The z-transform of a sequence π_k, $k = 0,1,...$, is defined as

$$P(z) = \sum_{k=0}^{\infty} \pi_k z^k, \tag{20}$$

Where z is complex variable and is such that P(z) is analytic i.e., $\sum_{k=0}^{\infty} \pi_k z^k < \infty$, in[2].

$$P(z) = \pi_0 + \pi_1 z = \frac{1 + (\rho_1 + \rho_2\alpha)z}{1 + \rho_1 + \rho_2\alpha} \tag{21}$$

Let N be the random variable that describes the number of customers in the system. The mean number of costumers:

$$E[N] = \frac{d}{dz}P(z)\bigg|_{z=1} = \frac{\rho_1 + \rho_2\alpha}{1 + \rho_1 + \rho_2\alpha}. \tag{22}$$

Second and higher moments may be computed from correspondingly higher derivatives. The k^{th} derivative of the z −transform evaluated at $z = 1$ gives the k^{th} factorial moment:

$$\lim_{z\to 1} P^{(k)}(z) = E[N(N-1)\ldots(N-k+1)]. \qquad (23)$$

Thus,

$$Var(N) = E([N(N+1)] + E[N] - E^2[N])$$
$$= \frac{\rho_1 + \rho_2\alpha}{(1+\rho_1+\rho_2\alpha)^2}\cdot \qquad (24)$$

3.2. Coxian Queue Using Laplace Transform

Let W be the random variable that describes waiting time of customers in the system. Laplace transform of W

$$\mathcal{L}_W(s) = \frac{(1-\alpha)\mu_1}{\mu_1+s} + \frac{\alpha\mu_1}{\mu_1+s}\frac{\mu_2}{\mu_2+s}\cdot \qquad (25)$$

Mean waiting time in system of a customerfor *Cox*(2) is found by formula (20).

$$E[W] = \frac{\mu_2 + \alpha\mu_1}{\mu_1\mu_2}\cdot \qquad (26)$$

By using the formula (19) are also obtained other moments. For example,

$$Var[W] = \frac{\mu_2^2 + \alpha\mu_1^2(2-\alpha)}{\mu_1^2\mu_2^2}\cdot \qquad (27)$$

3.3. The Optimization of Measures of Performance

Theorem 1.

$E[N]$ is maximum for $\alpha = 1$.

Proof. . We can rewrite equation (22) as following:

$$\frac{1}{E[N]} = 1 + \frac{1}{\rho_1+\rho_2\alpha}\cdot \qquad 0 \le \alpha \le 1 \qquad (28)$$

The minimum value of $1/E[N]$ is the maximum value of $E[N]$. $1/E[N]$is minimum for $\alpha = 1$. In other words,the mean number of costumers is maximum with probability $\alpha = 1$.

$$\max_{0\le\alpha\le 1} E[N] = \frac{\rho_1+\rho_2}{1+\rho_1+\rho_2}\cdot \qquad (29)$$

Loss probability

Let Π_{loss} be the loss probability of customer in the system. In this regards, since there is no queue in the system, loss probability is calculated as following

$$\Pi_{loss} = \pi_{01} + \pi_{10} = 1 - \pi_{00} \qquad (30)$$

$$\Pi_{loss} = 1 - \frac{1}{1+\rho_1+\rho_2\alpha} \qquad (30)'$$

3.4. Optimal Order of Servers

Considering $0 \le \alpha \le 1$, let μ_1 denotes the service parameter of the first phase and the second phase andμ_2 denotes theservice parameter of second phase. Indication of loss probabilities and measure of performance for $\mu_1 \ge \mu_2$ are illustrated with **(1)**. Similarly indication of loss probabilities and measure of performance for $\mu_1 \le \mu_2$ are illustrated with **(2)**.

By using the formula (24), we write $\Pi_{loss}^{(1)}$ and $\Pi_{loss}^{(2)}$ as follows:

$$\Pi_{loss}^{(1)} = \frac{\rho_1+\rho_2\alpha}{1+\rho_1+\rho_2\alpha} \qquad (31)$$

$$\Pi_{loss}^{(2)} = \frac{\rho_2+\rho_1\alpha}{1+\rho_2+\rho_1\alpha} \qquad (32)$$

The following theorem is given for optimal ordering.

Theorem 2.

In this system, if the parameter of the first phase greater than the parameter of the second phase,

$$\Pi_{loss}^{(1)} \le \Pi_{loss}^{(2)}. \qquad (33)$$

$$\text{Proof} .\mu_2 \le \mu_1 \qquad (34)$$

This inequalityisrewritten as following

$$\frac{1}{\mu_1} \le \frac{1}{\mu_2} \qquad (35)$$

$$\rho_1 \le \rho_2 \qquad (36)$$

$$\rho_1(1-\alpha) \le \rho_2(1-\alpha) \qquad (37)$$

$$\rho_1 + \rho_2\,\alpha \le \rho_2 + \rho_1\alpha \qquad (38)$$

$$\frac{1}{\rho_2+\rho_1\alpha} \le \frac{1}{\rho_1+\rho_2\alpha} \qquad (39)$$

$$1 + \frac{1}{\rho_2+\rho_1\alpha} \le 1 + \frac{1}{\rho_1+\rho_2\alpha} \qquad (40)$$

$$\left[\frac{1+\rho_1+\rho_2\,\alpha}{\rho_1+\rho_2\,\alpha}\right]^{-1} \le \left[\frac{1+\rho_2+\rho_1\,\alpha}{\rho_2+\rho_1\,\alpha}\right]^{-1} \qquad (41)$$

3.5. Queueing models derived from $M/Cox(2)/1/$ 0

The $M/Cox(2)/1/$ 0 stochastic model is illustrated in figure1.

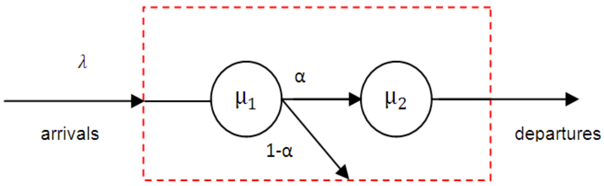

Figure 1.*Coxianqueueing model with two phases.*

In $M/Cox(2)\ /1\ /\ 0$ stochastic model if $\alpha = 1$ and $\mu_1 = \mu_2$ chosen then $M\ /\ E_2\ /\ 1\ /\ 0$ queueing system is obtained and this system is illustrated in figure 2.

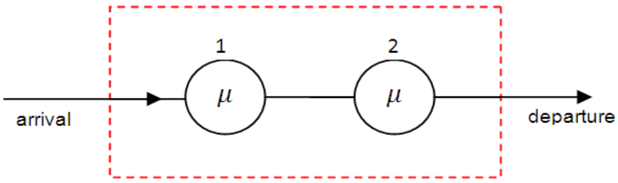

Figure 2. *M / E_2 / 1 / 0 queueing model.*

In $M/Cox(2)/1/\ 0$ queueing model if $\alpha = 1$ and $\mu_1 \neq \mu_2$ then $M\ /Hyp(\ 2)\ /\ 1\ /\ 0$ is obtained and this queueing model is illustrated in figure 3

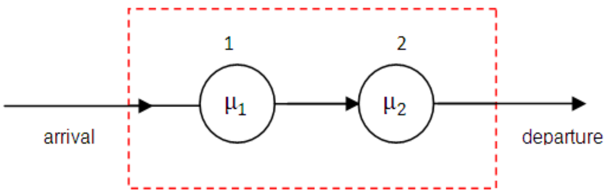

Figure 3. *The. M /Hyp(2) / 1 / 0 queuing model.*

Finally the $M/Cox(2)/1/\ 0$ model turns into the system $M/M/1/0$ for $\alpha = 0$ and this system is illustrated in figure4.

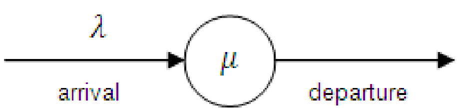

Figure 4.*The $M/M/1/0$ queueing model.*

As illustrated above, 3 different queueing models derived from $M/Cox(2)/1/\ 0$ system. There is no waiting at all these queueing models.

4. Numerical Example

In an interview consists of two juries, the arrivals of the candidates are according to Poisson stream with the parameter $\lambda = 2{,}3,$ no waiting allowed and the service parameters are $\mu_1 = 5{,}2$ and $\mu_2 = 3{,}2$ respectively according to Coxian distribution. For different values of α, loss probabilities and measure of performances are calculated and given in Table 1. When the juries' places in interview are changed (changing the parameters places) loss probabilities and measureof performances are calculated as well in Table 2. Assuming that $\mu_1 = \mu_2 = 4{,}2,$ loss probabilities and measure of performances are obtained in Table 3.

Table 1. *For $\lambda = 2.3,\ \mu_1 = 5{,}2\ ;\ \mu_2 = 3{,}2.$*

α	$\Pi_{loss}^{(1)}$	$Max\{E(N)\}$	$Var^{(1)}(N)$	$E^{(1)}(W)$	$Var^{(1)}(W)$
0,0	0,278	0,513	0,201	0,172	0,029
0,4	0,395	0,513	0,239	0,291	0,087
0,6	0,440	0,513	0,246	0,351	0,104
1,0	0,513	0,513	0,250	0,470	0,119

Table 2._For $\lambda = 2.3,\ \mu_1 = 3{,}2\ ;\ \mu_2 = 5{,}2.$_

α	$\Pi_{loss}^{(2)}$	$Max\{E(N)\}$	$Var^{(2)}\ (N)$	$E^{(2)}(W)$	$Var^{(2)}(W)$
0,0	0,401	0,513	0,240	0,299	0,089
0,4	0,451	0,513	0,248	0,367	0,108
0,6	0,474	0,513	0,249	0,402	0,114
1,0	0,513	0,513	0,250	0,470	0,119

Table 3._For $\lambda = 2.3,\ \mu_1 = \mu_2 = 4{,}2.$_

α	Π_{loss}	$Max\{E(N)\}$	$Var(N)$	$E(W)$	$Var(W)$
0,0	0.353	0.522	0.228	0.238	0.056
0,4	0.433	0.522	0.245	0.333	0.092
0,6	0.466	0.522	0.248	0.380	0.104
1,0	0.522	0.522	0.249	0.476	0.113

5. Conclusion

By analyzing this system, limit probabilities and probability mass function are obtained using generating function. The mean number of customer in the system and the loss probability of any customer are calculated. The optimization of the mean number of customer in the system is proved by Theorem 1: For both ordering, the optimal mean customer number in the system which is $Max\{E(N)\}$ is found to be the same value. The optimal ordering of phases for service parameters is found. It is observed that the first order is optimal according to measures of performances. The variance and Laplace transform of mean waiting time of a customer in system are given: As the mean service time is minimum for $\alpha = 0$, variance of service time is minimum for $\alpha = 1$. A numerical example is given on the subject. Furthermore it is shown that this analyzed system turns into different queueing systems when concerning the probability of customer's leaving the system and service parameters are same or different. The system *M / Cox(2)* / 1 / 0 for $\alpha = 1$ and $\mu_1 = \mu_2$ turns into the system $M\ /\ E_2\ /\ 0$: This situation is numerically shown. The system *M /Cox(2)/* 1/ 0 for $\alpha = 1$ and $\mu_1 \neq \mu_2$turns into the system $M\ /Hyp(\ 2)/\ 1$, in other words service time has hypo-exponential distribution and this situation is numerically shown. The system *M / Cox(2)*/ 1/0 for $\alpha = 0$ turns into the system $M\ /\ M\ /\ 1/\ 0$: Thus situations are numerically shown. For $\alpha < 1$, depending on the order it is clear that mean waiting time, the variance of waiting time and the variance of customer number in the system is less in first order. For later studies, increasing the number of phases may be advisable.

References

[1] D.R. Cox, "A use of complex probabilities in the theory of stochastic processes. Proc. Camb. Phil. Soc.,"313-19, 1955.

[2] W. J. Stewart, "Probability, Markov Chains, Queues, Simulation, New Jersey,"2009.

[3] F.Neuts, "Probability distributions of phase type, in Liber Amicorum Professor Emeritus H.Florin, Department of Mathematics", University of Louvain, Louvain, Belgium 173-206.

[4] U. N. Bhat, "An Introduction to Queuing Theory," Boston, 2008.

[5] R.Marie, "Calculating Equilibrim Probabilities for $\lambda(n)/C_k/1/N$ Queues," ACM Sigmetrics Performance Evaluation Review, Vol. 9, No.2, 1980.

[6] M. Zobu, V. Sağlam, M. Sağır, E. Yücesoy and T. Zaman"The Simulation and Minimization of Loss Probability in theTandem Queueing with Two Heterogeneous Channels," Mathematical Problems in Engineering, vol. 2013,Article ID 529010, 4, pages, 2013.

[7] M.Zobu "Using Sequential Analysis for Hypothesis Tests in The Phase-Type Distribution Queueing Systems," Ph.D. Thesis, OndokuzMayis University, Samsun, Turkey, 2013.

[8] D. Gross, C.M. Harris, "Fundamentals of Queuing Theory," Third Edition, John Wiley & Sons, New York, 1998.