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Numerical approximatons for solving partial differentıal equations with variable coefficients

Veyis TURUT

Department of Mathematics, Faculty of Arts and Sciences, Batman University, Batman,Turkey

Email address:
veyisturut@gmail.com (V. TURUT)

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Abstract: In this paper, variational iteration method (VIM) and multivariate padé approximaton (MPA) were compared. First, partial differential eqaution has been solved and converted to power series by variational iteration method (VIM), then the numerical solution of partial differential eqauation was put into multivariate padé series. Thus the numerical solutions of the partial differential eqautions were obtained. Numerical solutions of two examples were calculated and results were presented in tables and figures.

Keywords: Variational Iteration Method (VIM), Multivariate Padé Approximaton (MPA), Partial Differential Equation (PDE)

1. Introduction

Many powerful numerical and analytical methods have been presented. Among them, the Adomian decomposition method (ADM) [1-4], the variational iteration method (VIM) [5–8], differential transform method (DTM) and multivariate padé approximaton (MPA) [9-15] are relatively new approaches providing an analytical and numerical approximation to linear and nonlinear problems.

The variational iterational method (VIM) was first proposed by He [16,17] and has been succesfully applied to autonomous differential equations, non-linear partial differential equations, non-linear polycrystalline solids, and other fields.

Multivariate padé approximaton (MPA) has been successfully applied to solve partial differential equations. Many definitions and theorems have been developed for Multivariate Padé Approximations (MPA) (see [18] for a survey on Multivariate Padé approximation).

2. The Variational Iteration Method

The basic concepts and principles variational iteration method can be seen in [19-22]. Ali and Raslan [25] obtained the following iteration formula for general PDE equation (1) by using the basic concepts and principles of variational iteration method:

$$L_t u + L_x u + L_y u + L_z u + Nu = g(x,y,z,t). \tag{1}$$

where, L_t, L_x, L_y and L_z are linear operators of t, x, y and z, respectively, and N is a non-linear operator. According to VIM, the following correction functional can be expressed in t-, x-, y- and z-directions, respectively, as follows [25] :

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^t \lambda_1 \left\{ L_s u_n + (L_x + L_y + L_z + N)\tilde{u}_n - g \right\} ds, \tag{2}$$

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^x \lambda_2 \left\{ L_s u_n + (L_t + L_y + L_z + N)\tilde{u}_n - g \right\} ds, \tag{3}$$

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^y \lambda_3 \left\{ L_s u_n + (L_x + L_t + L_z + N)\tilde{u}_n - g \right\} ds, \tag{4}$$

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^z \lambda_4 \left\{ L_s u_n + (L_x + L_y + L_t + N)\tilde{u}_n - g \right\} ds. \tag{5}$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are general Lagrange multipliers [21], which can be identified optimally via the variational theory [21, 23] and $\tilde{u}_n$ is a restricted variaton which means

$\delta\tilde{u}_n = 0$. By this method, first the Lagrange multipliers are determined $\lambda_i (i = 1, 2, 3, 4)$ which will be identified optimally. The succesive approximations $u_{n+1}, n \geq 0$, of the solution u will be readily obtained by suitable choice of trial function u_0 [25]. Consequently , the correction functional will give several approximations. Then, one of this approximations was compared with multivariate padé approximation by putting into multivariate padé series .

3. Multivariate Padé Approximation

Consider the bivariate function $f(x, y)$ with Taylor series development

$$f(x,y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j \tag{6}$$

around the origin[24]. We know that a solution of unvariate Padé approximation problem for

$$f(x) = \sum_{i=0}^{\infty} c_i x^i \tag{7}$$

is given by

$$p(x) = \begin{vmatrix} \sum_{i=0}^{m} c_i x^i & x \sum_{i=0}^{m-1} c_i x^i & \cdots & x^n \sum_{i=0}^{m-n} c_i x^i \\ c_{m+1} & c_m & \cdots & c_{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{vmatrix} \tag{8}$$

and

$$q(x) = \begin{vmatrix} 1 & x & \cdots & x^n \\ c_{m+1} & c_m & \cdots & c_{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{vmatrix} \tag{9}$$

Let us now multiply j th row in $p(x)$ and $q(x)$ by x^{j+m-1} $(j = 2, \ldots, n+1)$ and afterwards divide j th column in $p(x)$ and $q(x)$ by x^{j-1} $(j = 2, \ldots, n+1)$. This results in a multiplication of numerator and denominator by x^{mn}. Having done so, we get

$$\frac{p(x)}{q(x)} = \frac{\begin{vmatrix} \sum_{i=0}^{m} c_i x^i & \sum_{i=0}^{m-1} c_i x^i & \cdots & \sum_{i=0}^{m-n} c_i x^i \\ c_{m+1} x^{m+1} & c_m x^m & \cdots & c_{m+1-n} x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n} x^{m+n} & c_{m+n-1} x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_{m+1} x^{m+1} & c_m x^m & \cdots & c_{m+1-n} x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n} x^{m+n} & c_{m+n-1} x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}} \tag{10}$$

if $(D = \det D_{m,n} \neq 0)$.

This quotent of determinants can also immediately be written down for a bivariate function $f(x, y)$. The sum $\sum_{i=0}^{k} c_i x^i$ shall be replaced k th partial sum of the Taylor series development of $f(x, y)$ and the expression $c_k x^k$ by an expression that contains all the terms of degree k in $f(x, y)$. Here a bivariate term $c_{ij} x^i y^j$ is said to be of degree $i + j$. If we define

$$p(x,y) = \begin{vmatrix} \sum_{i+j=0}^{m} c_{ij} x^i y^j & \sum_{i+j=0}^{m-1} c_{ij} x^i y^j & \cdots & \sum_{i+j=0}^{m-n} c_{ij} x^i y^j \\ \sum_{i+j=m+1} c_{ij} x^i y^j & \sum_{i+j=m} c_{ij} x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij} x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij} x^i y^j & \sum_{i+j=m+n-1} c_{ij} x^i y^j & \cdots & \sum_{i+j=m} c_{ij} x^i y^j \end{vmatrix} \tag{11}$$

and

$$q(x,y) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum_{i+j=m+1} c_{ij} x^i y^j & \sum_{i+j=m} c_{ij} x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij} x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij} x^i y^j & \sum_{i+j=m+n-1} c_{ij} x^i y^j & \cdots & \sum_{i+j=m} c_{ij} x^i y^j \end{vmatrix} \tag{12}$$

Then it is easy to see that $p(x, y)$ and $q(x, y)$ are of the form

$$\begin{aligned} p(x,y) &= \sum_{i+j=mn}^{mn+m} a_{ij} x^i y^j \\ q(x,y) &= \sum_{i+j=mn}^{mn+n} b_{ij} x^i y^j \end{aligned} \tag{13}$$

We know that $p(x, y)$ and $q(x, y)$ are called Padé equations[24]. So the multivariate Padé approximant of order (m, n) for $f(x, y)$ is defined as

$$r_{m,n}(x,y) = \frac{p(x,y)}{q(x,y)} \tag{14}$$

4. Applications and Results

In this section, the two methods VIM and MPA wil be illustrated by two examples. All the results are calculated by using software mapple.

Example 4.1.

Consider the one-dimensional heat equation with variable coefficients

$$u_t(x,t) - \frac{x^2}{2} u_{xx}(x,t) = 0, \tag{15}$$

and the initial condition $u(x, 0) = x^2$. The variational iterational schema of equation (15) has the form[25]

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{ (u_n(x,s))_s - \frac{x^2}{2} (\tilde{u}_n(x,s))_{xx} \right\} ds, \tag{16}$$

where $n \geq 0$ and $u_0(x, t) = x^2$. This yields the stationary conditions

$$1+\lambda|_{s=t}=0\,,\ \lambda'(s)=0. \tag{17}$$

Hence, the Lagrange multiplier is

$$\lambda=-1. \tag{18}$$

Substituting this value of the Lagrange multiplier into the functional (16) gives the iteration formula[25]

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left\{(u_n(x,s))_s-\frac{x^2}{2}(u_n(x,s))_{xx}\right\}ds. \tag{19}$$

Ali and Raslan obtained [25] the folowing succesive approximations, starting with an initial approximation:

$$u_0(x,t)=u(x,0)=x^2$$

and using the iteration formula (19),

$$u_1(x,t)=(1+t)x^2,$$

$$u_2(x,t)=(1+t+\frac{t^2}{2!})x^2,$$

$$u_3(x,t)=(1+t+\frac{t^2}{2!}+\frac{t^3}{3!})x^2,$$

$$u_4(x,t)=(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!})x^2,$$

$$u_5(x,t)=(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!})x^2,$$

$$u_6(x,t)=(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}+\frac{t^6}{6!})x^2. \tag{20}$$

The exact solution is given as $u(x,t)=e^t x^2$ in [28]. Now let us calculate the approximate solution of Eq.(20) for $m=6$ and $n=2$ by using Multivariate Padé approximation. To obtain Multivariate Padé equations of Eq.(20) for $m=6$ and $n=2$, we use Eqs.(11) and (12). By using Eqs.(11) and (12) We obtain,

$$p(x,t)=\begin{vmatrix} x^2+x^2t+\frac{1}{2}x^2t^2+\frac{1}{6}x^2t^3+\frac{1}{24}x^2t^4 & x^2+x^2t+\frac{1}{2}x^2t^2+\frac{1}{6}x^2t^3 & x^2+x^2t+\frac{1}{2}x^2t^2 \\ \frac{1}{120}x^2t^5 & \frac{1}{24}x^2t^4 & \frac{1}{6}x^2t^3 \\ \frac{1}{720}x^2t^6 & \frac{1}{120}x^2t^5 & \frac{1}{24}x^2t^4 \end{vmatrix}$$
$$=\frac{t^8(t^4+12t^3+72t^2+240t+360)x^6}{1036800}$$

and

$$q(x,t)=\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{120}x^2t^5 & \frac{1}{24}x^2t^4 & \frac{1}{6}x^2t^3 \\ \frac{1}{720}x^2t^6 & \frac{1}{120}x^2t^5 & \frac{1}{24}x^2t^4 \end{vmatrix}=$$
$$\frac{t^8(30-10t+t^2)x^4}{86400}$$

So the Multivariate Padé approximation of order $(6,2)$ for eq.(20), that is

$$r_{6,2}(x,t)=\frac{(t^4+12t^3+72t^2+240t+360)x^2}{12(30-10t+t^2)} \tag{21}$$

Example 4.2.

Consider the one-dimensional wave equation with variable coefficients

$$u_{tt}(x,t)-\frac{x^2}{2}u_{xx}(x,t)=0, \tag{22}$$

with initial conditions $u(x,0)=x$, $u_t(x,0)=x^2$. The correction functional for equation (22) is given as[25]

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda\left\{(u_n(x,s))_{ss}-\frac{x^2}{2}(\tilde{u}_n(x,s))_{xx}\right\}ds, \tag{23}$$

where $n\geq 0$ and $u_0(x,t)=x+tx^2$. Then, the stationary conditions are

$$1+\lambda'|_{s=t}=0,\quad 1+\lambda|_{s=t}=0,\quad \lambda''(s)=0. \tag{24}$$

This in turn gives

$$\lambda=s-t \tag{25}$$

Substituting this value of the Lagrange multiplier into functional (23) gives the iteration formula [25]

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t(s-t)\left\{(u_n(x,s))_{ss}-\frac{x^2}{2}(u_n(x,s))_{xx}\right\}ds. \tag{26}$$

Selecting the initial approximation: $u_0(x,t)=x+tx^2$, with the itration Formula (26), Ali and Raslan obtained [25] the following succesive approximations

$$\begin{aligned} u_1(x,t)&=x+(t+\frac{t^3}{3!})x^2,\\ u_2(x,t)&=x+(t+\frac{t^3}{3!}+\frac{t^5}{5!})x^2,\\ u_3(x,t)&=x+(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!})x^2,\\ u_4(x,t)&=x+(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+\frac{t^9}{9!})x^2, \end{aligned} \tag{27}$$

$$u_5(x,t)=x+(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+\frac{t^9}{9!}+\frac{t^{11}}{11!})x^2,$$

The exact solution is given as $u(x,t)=x+x^2\sinh t$ in [28]. Now let us calculate the approximate solution of Eq.(20) for $m=11$ and $n=2$ by using Multivariate Padé approximation. To obtain Multivariate Padé equations of Eq.(20) for $m=11$ and $n=2$, we use Eqs.(11) and (12). By using Eqs.(11) and (12) We obtain

$$p(x,t)=\begin{vmatrix} x+x^2t+\frac{1}{6}x^2t^3+\frac{1}{120}x^2t^5+\frac{1}{5040}x^2t^7+\frac{1}{362880}x^2t^9 & x+x^2t+\frac{1}{6}x^2t^3+\frac{1}{120}x^2t^5+\frac{1}{5040}x^2t^7 & x+x^2t+\frac{1}{6}x^2t^3+\frac{1}{120}x^2t^5+\frac{1}{5040}x^2t^7 \\ 0 & \frac{1}{362880}x^2t^9 & 0 \\ \frac{1}{39916800}x^2t^{11} & 0 & \frac{1}{362880}x^2t^9 \end{vmatrix}$$
$$=\frac{(-181440t^2+3144960xt^3+136080xt^5+2448xt^7+19xt^9+19958400xt+19958400)x^5t^{18}}{262815992119290000}$$

and

$$q(x,t)=\begin{vmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{362880}x^2t^9 & 0 \\ \frac{1}{39916800}x^2t^{11} & 0 & \frac{1}{362880}x^2t^9 \end{vmatrix}$$
$$=-\frac{(-110+t^2)x^4t^{18}}{14485008384000}$$

So the Multivariate Padé approximation of order $(11,2)$ for eq.(27), that is

$$r_{11,2}(x,t)=-\frac{(-181440t^2+3144960xt^3+136080xt^5+2448xt^7+19xt^9+19958400xt+19958400)x}{(181440(-110t^2))} \quad (28)$$

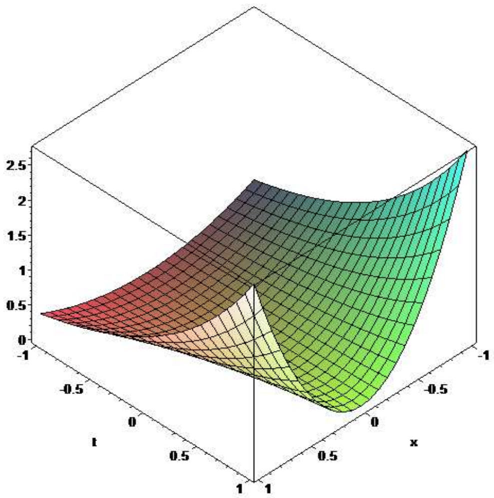

Figure 1. *Exact solution of partial differential equation in example 1.*

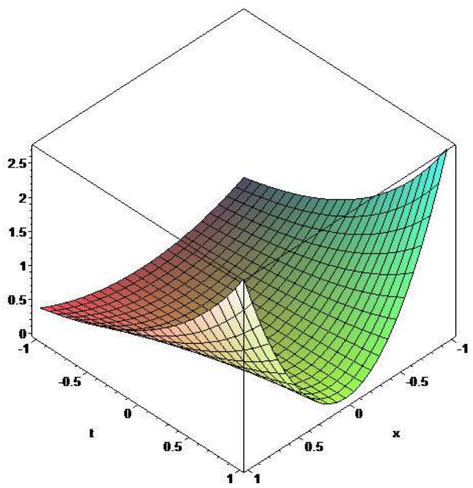

Figure 2. *Multivariate Padé approximation for VIM solution of partial differential equation Example 1.*

Table 1. *Comparison of VIM and MPA for example 1.*

x	t	Exact solution	Approximate solution withMPA	Absolute error of MPA
1.0	1.0	2.718281828	2.718253968	0.000027860
0.9	0.9	1.992278520	1.992268535	0.9985×10^{-5}
0.8	0.8	1.424346194	1.424342992	0.3202×10^{-5}
0.7	0.7	0.9867388264	0.9867379342	0.8922×10^{-6}
0.6	0.6	0.6559627680	0.6559625616	0.2064×10^{-6}
0.5	0.5	0.4121803178	0.4121802805	0.373×10^{-7}
0.4	0.4	0.2386919517	0.2386919470	0.47×10^{-8}
0.3	0.3	0.1214872927	0.1214872923	0.4×10^{-9}
0.2	0.2	0.04885611032	0.04885611032	0.0
0.1	0.1	0.01105170918	0.01105170918	0.0

Table 2. *Comparison of VIM and MPA for example 2.*

x	t	Exact solution	Approximate solution with-MPA	Absolute error of MPA
-2.0	-2.0	12.50744163	12.50744401	0.238×10^{-5}
-1.9	-1.9	9.89806811	9.898069207	0.110×10^{-5}
-1.8	-1.8	7.732644693	7.732645177	0.484×10^{-6}
-1.7	1.7	5.945876289	5.945876494	0.205×10^{-6}
-1.6	-1.6	4.481453960	4.481454042	0.82×10^{-7}
-1.5	-1.5	3.290878774	3.290878805	0.31×10^{-7}
-1.4	-1.4	2.332430942	2.332430954	0.12×10^{-7}
-1.3	-1.3	1.570266319	1.570266322	0.3×10^{-8}
-1.2	-1.2	0.973624351	0.9736243524	0.1×10^{-8}
-1.1	-1.1	0.516133439	0.5161334388	0.0
-1.0	-1.0	0.175201194	0.1752011937	0.0

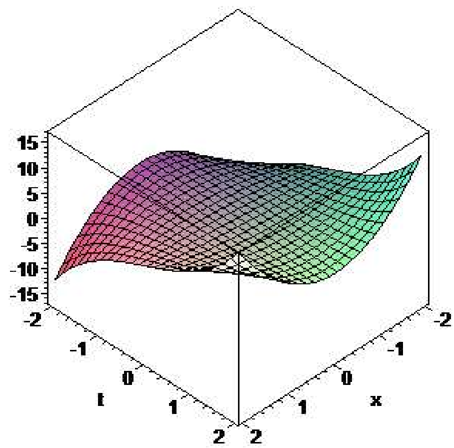

Figure 3. *Exact solution of partial differential equation in example 2.*

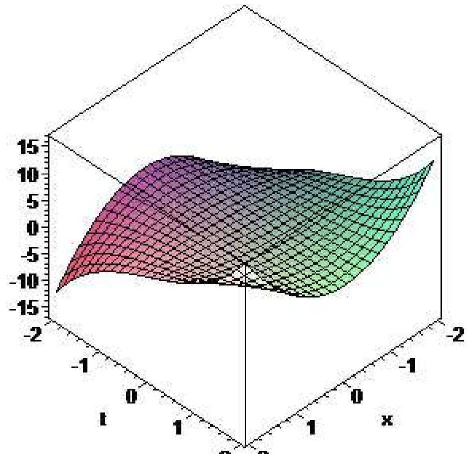

Figure 4. *Multivariate Padé approximation for VIM solution of partial differential equation in Example2.*

5. Conclusion

The figure which is obtained using MPA and the figure of the exact solution in three-dimensional are shown in figure (1-2) and figure (3-4). As can be seen in table 1, table 2 and figure (1-2), figure (3-4), the approximation solutions with MPA are quite close to exact solutions. It is also observed that MPA is robust and applicable to various types of partial differential equations.

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